# The conformal geometry
## of
# Random Regge Triangulations

*M. Carfora* [b,1],

*C. Dappiaggi* [b,2],

*A. Marzuoli* [b,3],

[b] Dipartimento di Fisica Nucleare e Teorica,
Università degli Studi di Pavia,
via A. Bassi 6, I-27100 Pavia, Italy,
and
Istituto Nazionale di Fisica Nucleare, Sezione di Pavia,
via A. Bassi 6, I-27100 Pavia, Italy

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[1]email mauro.carfora@pv.infn.it
[2]email claudio.dappiaggi@pv.infn.it
[3]email annalisa.marzuoli@pv.infn.it

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# 1 Introduction

We present here a report on our efforts to understand the conformal geometry associated with the use of simplicial methods in 2-dimensional quantum gravity. In particular, we trace a logical path among the main characters of such a theory: dynamical triangulations, Regge surfaces, and Riemann moduli theory. The choice of such a subject seems appropriate in a volume of papers dedicated to A. Lichnerowicz, since the interplay between conformal geometry and gravity has always been an important motivation in his research. Such an interplay is crucial in 2-dimensional (quantum) gravity, and it is safe to say that the successful analysis of such a theory can be traced back to the special role that conformal geometry plays in dimension two. However, when we enter a region of strong coupling regime between matter and gravity, we do not yet have a complete understanding of the field-theoretic dynamics of the conformal mode of the theory. This is an important problem whose relevance goes far beyond the study of two-dimensional gravity and string theories and has received considerable attention in recent years. While we are still far from a satisfactory solution, it is fair to observe that some clue to its understanding are coming out from the use of techniques of Piecewise-Linear (PL) geometry. As a matter of fact, triangulated surfaces provide one of the most powerful techniques for analyzing two-dimensional quantum gravity in regimes which are not accessible to the standard field-theoretic formalism. In such a sense, it is important to establish a connection between conformal geometry and quantum gravity which is more directly related with the use of simplicial methods, thus making their role in the theory more explicit.

In this paper, which is a natural evolution of one of our previous works [1], we discuss such a topic by using surfaces endowed with triangulations with variable connectivity and fluctuating egde-lengths. Such piecewise-linear surfaces are not proper Regge triangulations, since their adjacency matrix is not a priori fixed, nor they are dynamical triangulations, since they are generated by glueing triangles which are not, in general, equilateral. Random Regge Triangulations seems an appropriate name [1]. They allow for a relatively simple and direct analyisis of the modular properties of 2 dimensional simplicial quantum gravity, since such triangulations are naturally related to the Weil-Petersson geometry of the (compactified) moduli space of genus $g$ Riemann surfaces with $N_0$ punctures $\overline{\mathfrak{M}}_{g,N_0}$, (the number of punctures $N_0$ is the number of vertices of the triangulations). The main result of our analysis is the explicit association of a Weil-Petersson metric to a Regge triangulation. With such a metric at our disposal, we can formally evaluate the Weil-Petersson volume over the space of all random Regge triangulations. By exploiting a recent result of Manin and Zograf [2], we can show that such a volume provides the (large $N_0$ asymptotics of the) dynamical triangulation partition function for pure gravity. We conclude the paper by discussing the (regularized) Liouville action associated with random Regge triangulations and its connection with Hodge-Deligne theory. From a field theoretic point of view there is a tendency to relegate simplicial methods to the ancillary role of a regularization scheme, playing a role in gravity

similar to that of lattice regularizations in gauge theories. However, according to some of the results discussed in this paper, one cannot help thinking that in gravity their role is more foundational, and that they rely on a set of first principles probably connected with the holographic hypothesis [3],[4]

## 2   Triangulated surfaces and Ribbon graphs

Let $T$ denote a 2-dimensional simplicial complex with underlying polyhedron $|T|$ and $f$- vector $(N_0(T), N_1(T), N_2(T))$, where $N_i(T) \in \mathbb{N}$ is the number of $i$-dimensional sub- simplices $\sigma^i$ of $T$. Given a simplex $\sigma$ we denote by $st(\sigma)$, (the star of $\sigma$), the union of all simplices of which $\sigma$ is a face, and by $lk(\sigma)$, (the link of $\sigma$), is the union of all faces $\sigma^f$ of the simplices in $st(\sigma)$ such that $\sigma^f \cap \sigma = \emptyset$. A random Regge triangulation of a 2 -dimensional PL manifold $M$, (without boundary), is a homeomorphism $|T_l| \to M$ where each face of $T$ is geometrically realized by a rectilinear simplex of variable edge-lengths $l(\sigma^1(k))$. A dynamical triangulation $|T_{l=a}| \to M$ is a particular case of a random Regge PL-manifold realized by rectilinear and equilateral simplices of edge-length $l(\sigma^1(k)) = a$.
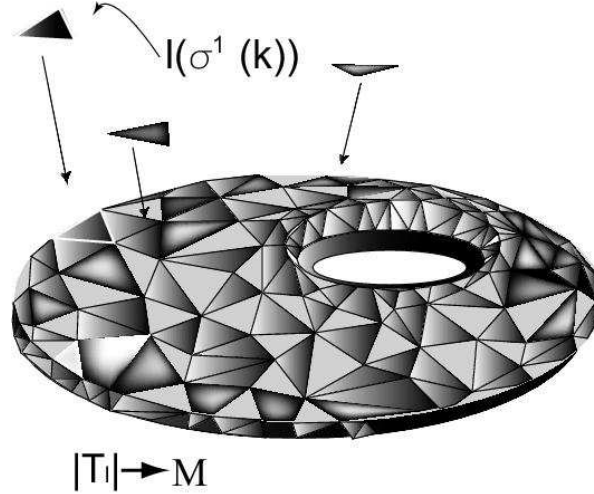


Figure 1: A torus triangulated with triangles of variable edge-length.

The metric structure of a Regge triangulation is locally Euclidean everywhere except at the vertices $\sigma^0$, (the *bones*), where the sum of the dihedral angles, $\theta(\sigma^2)$, of the incident triangles $\sigma^2$'s is in excess (negative curvature) or in defect (positive curvature) with respect to the $2\pi$ flatness constraint. The corresponding deficit angle $\varepsilon$ is defined by $\varepsilon = 2\pi - \sum_{\sigma^2} \theta(\sigma^2)$, where the sum-

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mation is extended to all 2-dimensional simplices incident on the given bone $\sigma^0$. If $K_T^0$ denotes the $(0)$-skeleton of $|T_l| \to M$, (*i.e.*, the collection of vertices of the triangulation), then $M \backslash K_T^0$ is a flat Riemannian manifold, and any point in the interior of an $r$- simplex $\sigma^r$ has a neighborhood homeomorphic to $B^r \times C(lk(\sigma^r))$, where $B^r$ denotes the ball in $\mathbb{R}^n$ and $C(lk(\sigma^r))$ is the cone over the link $lk(\sigma^r)$, (the product $lk(\sigma^r) \times [0,1]$ with $lk(\sigma^r) \times \{1\}$ identified to a point). In particular, let us denote by $C|lk(\sigma^0(k))|$ the cone over the link of the vertex $\sigma^0(k)$. On any such a disk $C|lk(\sigma^0(k))|$ we can introduce a locally uniformizing complex coordinate $\zeta(k) \in \mathbb{C}$ in terms of which we can explicitly write down a conformal conical metric locally characterizing the singular structure of $|T_l| \to M$, *viz.*,

$$e^{2u} \left| \zeta(k) - \zeta_k(\sigma^0(k)) \right|^{-2\left(\frac{\varepsilon(k)}{2\pi}\right)} |d\zeta(k)|^2 \,, \tag{1}$$

where $\varepsilon(k)$ is the corresponding deficit angle, and $u : B^2 \to \mathbb{R}$ is a continuous function ($C^2$ on $B^2 - \{\sigma^0(k)\}$) such that, for $\zeta(k) \to \zeta_k(\sigma^0(k))$, we have $\left| \zeta(k) - \zeta_k(\sigma^0(k)) \right| \frac{\partial u}{\partial \zeta(k)}$, and $\left| \zeta(k) - \zeta_k(\sigma^0(k)) \right| \frac{\partial u}{\partial \bar{\zeta}(k)}$ both $\to 0$. Up to the presence of $e^{2u}$, we immediately recognize in such an expression the metric $g_{\theta(k)}$ of a Euclidean cone of total angle $\theta(k) = 2\pi - \varepsilon(k)$. The factor $e^{2u}$ allows to move within the conformal class of all metrics possessing the same singular structure of the triangulated surface $|T_l| \to M$.



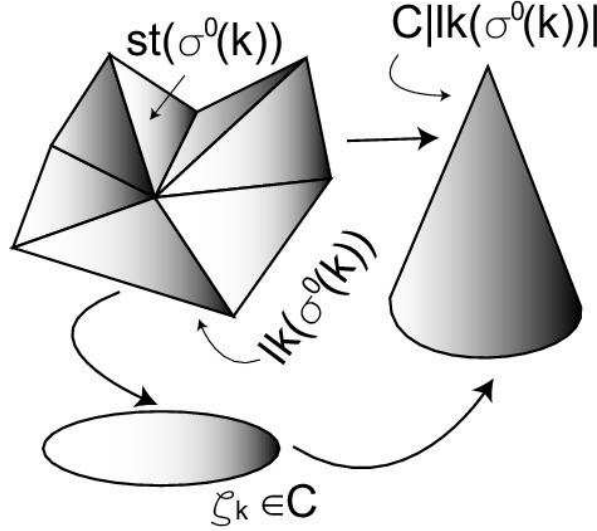Figure 2: The geometric structures around a vertex.

We shall denote by

$$\left[ g_{\theta(k)} \right] \doteq \left\{ e^{2u} g_{\theta(k)} \,:\, u \in C^0(B^2) \cap C^2(B^2 - \{\sigma^0(k)\}) \right\} \tag{2}$$

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the conformal class determined by the conical metric $g_{\theta(k)}$. If we allow for conformal factors with logarithmic singularities then we can equivalently consider the conformal class of the metric representing the Regge triangulation as given by $ds^2 = e^{2v}ds_0^2$ where $ds_0^2$ is a smooth metric on $M$ and where the conformal factor $v$ around the generic vertex $\sigma^0(k)$, is provided by

$$v|_{U_{\rho^2(k)}} = -\frac{\varepsilon(k)}{2\pi} \ln |\zeta(k) - \zeta_k(\sigma^0(k))| + u. \tag{3}$$

We can profitably shift between these two points of view by exploiting standard techniques of complex analysis, and making contact with moduli space theory.

### 2.1 Curvature assignments and divisors.

In the case of dynamical triangulations, the picture simplifies considerably since the deficit angles are generated by the numbers $\#\{\sigma^2(h) \perp \sigma^0(i)\}$ of triangles incident on the $N_0(T)$ vertices, the *curvature assignments*, $\{q(k)\}_{k=1}^{N_0(T)} \in \mathbb{N}^{N_0(T)}$,

$$q(i) = \frac{2\pi - \varepsilon(i)}{\arccos(1/2)}. \tag{4}$$

For a regular triangulation we have $q(k) \geq 3$, and since each triangle has 3 vertices $\sigma^0$, the set of integers $\{q(k)\}_{k=1}^{N_0(T)}$ is constrained by

$$\sum_k^{N_0} q(k) = 3N_2 = 6\left[1 - \frac{\chi(M)}{N_0(T)}\right] N_0(T), \tag{5}$$

where $\chi(M)$ denotes the Euler-Poincaré characteristic of the surface, and where $6\left[1 - \frac{\chi(M)}{N_0(T)}\right]$, ($\simeq 6$ for $N_0(T) >> 1$), is the average value of the curvature assignments $\{q(k)\}_{k=1}^{N_0}$. More generally we shall consider semi-simplicial complexes for which the constraint $q(k) \geq 3$ is removed. Examples of such configurations are afforded by triangulations with pockets, where two triangles are incident on a vertex, or by triangulations where the star of a vertex may contain just one triangle. We shall refer to such extended configurations as generalized (Regge and dynamical) triangulations.

The singular structure of the metric defined by (1) can be naturally summarized in a formal linear combination of the points $\{\sigma^0(k)\}$ with coefficients given by the corresponding deficit angles (normalized to $2\pi$), *viz.*, in the *real divisor* [5]

$$Div(T) \doteq \sum_{k=1}^{N_0(T)} \left(-\frac{\varepsilon(k)}{2\pi}\right) \sigma^0(k) = \sum_{k=1}^{N_0(T)} \left(\frac{\theta(k)}{2\pi} - 1\right) \sigma^0(k) \tag{6}$$

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supported on the set of bones $\{\sigma^0(i)\}_{i=1}^{N_0(T)}$. Note that the degree of such a divisor, defined by

$$|Div(T)| \doteq \sum_{k=1}^{N_0(T)} \left( \frac{\theta(k)}{2\pi} - 1 \right) = -\chi(M) \qquad (7)$$

is, for dynamical triangulations, a rewriting of the combinatorial constraint (5). In such a sense, the pair $(|T_{l=a}| \to M, Div(T))$, or shortly, $(T, Div(T))$, encodes the datum of the triangulation $|T_{l=a}| \to M$ and of a corresponding set of curvature assignments $\{q(k)\}$ on the vertices $\{\sigma^0(i)\}_{i=1}^{N_0(T)}$. The real divisor $|Div(T)|$ characterizes the Euler class of the pair $(T, Div(T))$ and yields for a corresponding Gauss-Bonnet formula. Explicitly, the Euler number associated with $(T, Div(T))$ is defined, [5], by

$$e(T, Div(T)) \doteq \chi(M) + |Div(T)|. \qquad (8)$$

and the Gauss-Bonnet formula reads [5]:

**Lemma 1** *(**Gauss-Bonnet for triangulated surfaces**) Let $(T, Div(T))$ be a triangulated surface with divisor*

$$Div(T) \doteq \sum_{k=1}^{N_0(T)} \left( \frac{\theta(k)}{2\pi} - 1 \right) \sigma^0(k), \qquad (9)$$

*associated with the vertices $\{\sigma^0(k)\}_{k=1}^{N_0(T)}$. Let $ds^2$ be the conformal metric (1) representing the divisor $Div(T)$ . Then*

$$\frac{1}{2\pi} \int_M K dA = e(T, Div(T)), \qquad (10)$$

*where $K$ and $dA$ respectively are the curvature and the area element corresponding to the local metric $ds^2_{(k)}$.*

Note that such a theorem holds for any singular Riemann surface $\Sigma$ described by a divisor $Div(\Sigma)$ and not just for triangulated surfaces [5]. Since for a Regge (dynamical) triangulation, we have $e(T_a, Div(T)) = 0$, the Gauss-Bonnet formula implies

$$\frac{1}{2\pi} \int_M K dA = 0. \qquad (11)$$

Thus, a triangulation $|T_l| \to M$ naturally carries a conformally flat structure. Clearly this is a rather obvious result, (since the metric in $M - \{\sigma^0(i)\}_{i=1}^{N_0(T)}$ is flat). However, it admits a not-trivial converse (recently proved by M. Troyanov, but, in a sense, going back to E. Picard) [5], [6]:

**Theorem 2** *(**Troyanov-Picard**) Let $((M, \mathcal{C}_{sg}), Div)$ be a singular Riemann surface with a divisor such that $e(M, Div) = 0$. Then there exists on $M$ a unique (up to homothety) conformally flat metric representing the divisor $Div$.*

## 2.2. Conical Regge polytopes.

Let us consider the (first) barycentric subdivision of $T$. The closed stars, in such a subdivision, of the vertices of the original triangulation $T_l$ form a collection of 2-cells $\{\rho^2(i)\}_{i=1}^{N_0(T)}$ characterizing the *conical* Regge polytope $|P_{T_l}| \to M$ (and its rigid equilateral specialization $|P_{T_a}| \to M$) barycentrically dual to $|T_l| \to M$. If $(\lambda(k), \chi(k))$ denote polar coordinates (based at $\sigma^0(k)$) of a point $p \in \rho^2(k)$, then $\rho^2(k)$ is geometrically realized as the space

$$\{(\lambda(k), \chi(k)) \; : \lambda(k) \geq 0; \chi(k) \in \mathbb{R}/(2\pi - \varepsilon(k))\mathbb{Z}\}/ (0, \chi(k)) \sim (0, \chi'(k)) \quad (12)$$

endowed with the metric

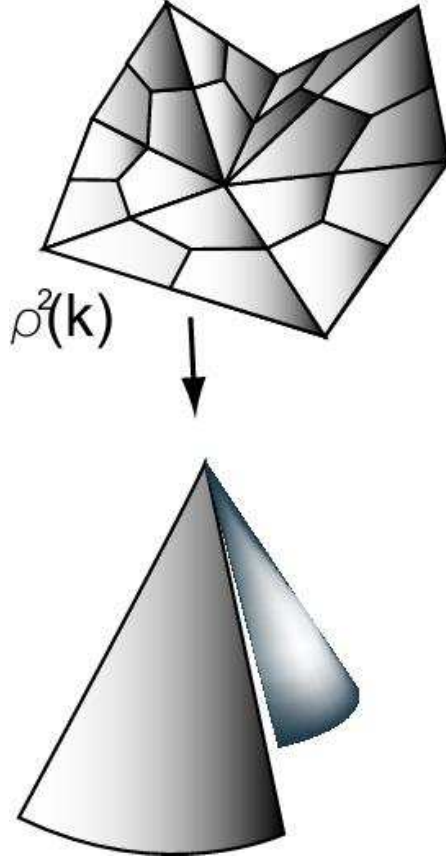$$d\lambda(k)^2 + \lambda(k)^2 d\chi(k)^2. \quad (13)$$



Figure 3: The conical geometry of the baricentrically dual polytope.

In other words, here we are not considering a rectilinear presentation of the

dual cell complex $P$ (where the PL-polytope is realized by flat polygonal 2-cells $\{\rho^2(i)\}_{i=1}^{N_0(T)}$) but rather a geometrical presentation $|P_{T_l}| \to M$ of $P$ where the 2 -cells $\{\rho^2(i)\}_{i=1}^{N_0(T)}$ retain the conical geometry induced on the barycentric subdivision by the original metric (1) structure of $|T_l| \to M$.

## 2.3. Hyperbolic cusps and cylindrical ends

It is important to stress that wheras a Regge triangulation characterizes a unique (up to automorphisms) singular Euclidean structure, this latter actually allows for a more general type of metric triangulation. The point is that some of the vertices associated with a singular Euclidean structure can be characterized by deficit angles $\varepsilon(k) \to 2\pi$, $\ i.e.,\ \sum_{\sigma^2(k)} \theta(\sigma^2(k)) = 0$, where the summation is extended to all triangles incident on the given vertex $\sigma^0(k) \in |T_l| \to M$. Such a situation corresponds to having the cone $C|lk(\sigma^0(k))|$ over the link $lk(\sigma^0(k))$ realized by a Euclidean cone of angle 0. This is a natural limiting case in a Regge triangulation, (think of a vertex where many long and thin triangles are incident), and it is usually discarded as an unwanted pathology. However, there is really nothing pathological about that. It can be easily handled, since the corresponding 2-cell $\rho^2(k) \in |P_{T_l}| \to M$ can be naturally endowed with the geometry of a hyperbolic cusp, $i.e.$, that of a half-infinite cylinder $\mathbb{S}^1 \times \mathbb{R}^+$ equipped with the hyperbolic metric $\lambda(k)^{-2}(d\lambda(k)^2 + d\chi(k)^2)$. The triangles incident on $\sigma^0(k)$ are then realized as hyperbolic triangles with the vertex $\sigma^0(k)$ located at $\lambda(k) = \infty$ and corresponding angle $\theta_k = 0$ [7]. Alternatively, and perhaps more in the spirit of Regge calculus, one may consider $\rho^2(k)$ endowed with the conformal Euclidean structure obtained from (1) by setting $\frac{\varepsilon(k)}{2\pi} = 1$, $i.e.$

$$e^{2u} \left| \zeta(k) - \zeta_k(\sigma^0(k)) \right|^{-2} |d\zeta(k)|^2 , \tag{14}$$

which (up to the conformal factor $e^{2u}$) is the flat metric on the half-infinite cylinder $\mathbb{S}^1 \times \mathbb{R}^+$ (a cylindrical end).
Since the Poincaré metric on the punctured disk

$$D^2(k) = \{\zeta(k) \in C| \ 0 < \left|\zeta(k) - \zeta_k(\sigma^0(k))\right| < 1\}$$

is

$$\left( \left|\zeta(k) - \zeta_k(\sigma^0(k))\right| \ln \frac{1}{|\zeta(k) - \zeta_k(\sigma^0(k))|} \right)^{-2} |d\zeta(k)|^2 , \tag{15}$$

one can shift from the Euclidean to the hyperbolic metric by setting

$$e^{2u} = \left( \ln \frac{1}{|\zeta(k) - \zeta_k(\sigma^0(k))|} \right)^{-2} , \tag{16}$$

and the two points of view are fully equivalent. At any rate the presence of such defects (hyperbolic cusps or cylindrical ends) is consistent with a singular
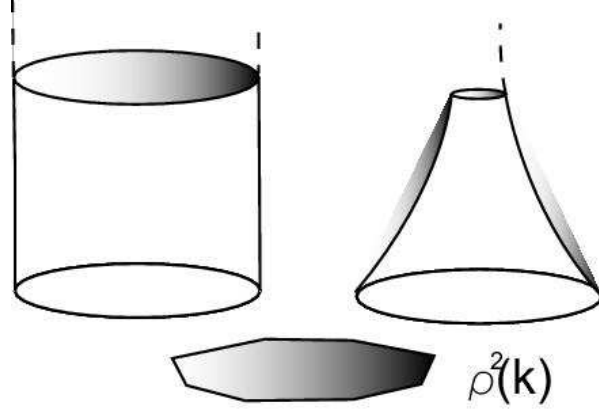
Figure 4: The cylindrical and hyperbolic metric over a $\theta \to 0$ degenerating polytopal cell.

Euclidean structure as long as the associated divisor satisfies the topological constraint (7), which we can rewrite as

$$\sum_{\{\frac{\varepsilon(k)}{2\pi} \neq 1\}} \left(-\frac{\varepsilon(k)}{2\pi}\right) = 2g - 2 + \#\left\{\sigma^0(h)|\ \frac{\varepsilon(h)}{2\pi} = 1\right\}. \tag{17}$$

In particular, we can have the limiting case of the singular Euclidean structure associated with a genus $g$ surface triangulated with $N_0 - 1$ hyperbolic vertices $\{\sigma^0(k)\}_{k=1}^{N_0-1}$ (or, equivalently, with $N_0 - 1$ cylindrical ends) and just one standard conical vertex, $\sigma^0(N_0)$, supporting the deficit angle

$$-\frac{\varepsilon(N_0)}{2\pi} = 2g - 2 + (N_0 - 1). \tag{18}$$

**2.4. Ribbon graphs.**

The geometrical realization of the 1-skeleton of the conical Regge polytope $|P_{T_l}| \to M$ is a 3-valent graph

$$\Gamma = (\{\rho^0(k)\}, \{\rho^1(j)\}) \tag{19}$$

where the vertex set $\{\rho^0(k)\}_{k=1}^{N_2(T)}$ is identified with the barycenters of the triangles $\{\sigma^o(k)\}_{k=1}^{N_2(T)} \in |T_l| \to M$, whereas each edge $\rho^1(j) \in \{\rho^1(j)\}_{j=1}^{N_1(T)}$ is generated by two half-edges $\rho^1(j)^+$ and $\rho^1(j)^-$ joined through the barycenters $\{W(h)\}_{h=1}^{N_1(T)}$ of the edges $\{\sigma^1(h)\}$ belonging to the original triangulation $|T_l| \to M$. If we formally introduce a ghost-vertex of a degree 2 at each middle point $\{W(h)\}_{h=1}^{N_1(T)}$ , then the actual graph naturally associated to the 1-skeleton

of $|P_{T_l}| \to M$ is the edge-refinement [8] of $\Gamma = (\{\rho^0(k)\}, \{\rho^1(j)\})$, *i.e.*

$$\Gamma_{ref} = \left( \{\rho^0(k)\} \bigsqcup_{h=1}^{N_1(T)} \{W(h)\}, \{\rho^1(j)^+\} \bigsqcup_{j=1}^{N_1(T)} \{\rho^1(j)^-\} \right). \qquad (20)$$

The natural automorphism group $Aut(P_l)$ of $|P_{T_l}| \to M$, (*i.e.*, the set of bijective maps $\Gamma = (\{\rho^0(k)\}, \{\rho^1(j)\}) \to \widetilde{\Gamma} = (\widehat{\{\rho^0(k)\}}, \widehat{\{\rho^1(j)\}})$ preserving the incidence relations defining the graph structure), is the automorphism group of its edge refinement [8], $Aut(P_l) \doteq Aut(\Gamma_{ref})$. The locally uniformizing complex coordinate $\zeta(k) \in \mathbb{C}$ in terms of which we can explicitly write down the singular Euclidean metric (1) around each vertex $\sigma^0(k) \in |T_l| \to M$, provides a (counterclockwise) orientation in the 2-cells of $|P_{T_l}| \to M$. Such an orientation gives rise to a cyclic ordering on the set of half-edges $\{\rho^1(j)^\pm\}_{j=1}^{N_1(T)}$ incident on the vertices $\{\rho^0(k)\}_{k=1}^{N_2(T)}$.
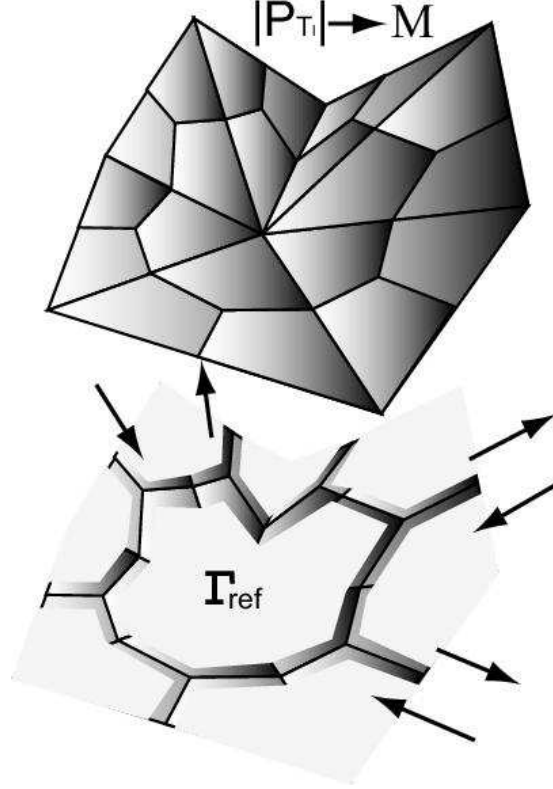


Figure 5: The dual polytope around a vertex and it's edge refinement.

According to these remarks, the 1-skeleton of $|P_{T_l}| \to M$ is a ribbon (or fat)

graph [9], *viz.*, a graph $\Gamma$ together with a cyclic ordering on the set of half-edges incident to each vertex of $\Gamma$. Conversely, any ribbon graph $\Gamma$ characterizes an oriented surface $M(\Gamma)$ with boundary possessing $\Gamma$ as a spine, (*i.e.*, the inclusion $\Gamma \hookrightarrow M(\Gamma)$ is a homotopy equivalence). In this way (the edge-refinement of) the 1-skeleton of a generalized conical Regge polytope $|P_{T_l}| \to M$ is in a one-to-one correspondence with trivalent metric ribbon graphs. The set of all such trivalent ribbon graph $\Gamma$ with given edge-set $e(\Gamma)$ can be characterized [8], [10] as a space homeomorphic to $\mathbb{R}_+^{|e(\Gamma)|}$, ($|e(\Gamma)|$ denoting the number of edges in $e(\Gamma)$), topologized by the standard $\epsilon$-neighborhoods $U_\epsilon \subset \mathbb{R}_+^{|e(\Gamma)|}$. The automorphism group $Aut(\Gamma)$ acts naturally on such a space via the homomorphism $Aut(\Gamma) \to \mathfrak{G}_{e(\Gamma)}$, where $\mathfrak{G}_{e(\Gamma)}$ denotes the symmetric group over $|e(\Gamma)|$ elements, and the resulting quotient space $\mathbb{R}_+^{|e(\Gamma)|}/Aut(\Gamma)$ is a differentiable orbifold.

## 2.5. The space of 1-skeletons of Regge polytopes.

Let $Aut_\partial(P_l) \subset Aut(P_l)$, denote the subgroup of ribbon graph automorphisms of the (trivalent) 1-skeleton $\Gamma$ of $|P_{T_l}| \to M$ that preserve the (labeling of the) boundary components of $\Gamma$. Then, the space $K_1 RP_{g,N_0}^{met}$ of 1-skeletons of conical Regge polytopes $|P_{T_l}| \to M$, with $N_0(T)$ labelled boundary components, on a surface $M$ of genus $g$ can be defined by [8]

$$K_1 RP_{g,N_0}^{met} = \bigsqcup_{\Gamma \in RGB_{g,N_0}} \frac{\mathbb{R}_+^{|e(\Gamma)|}}{Aut_\partial(P_l)}, \tag{21}$$

where the disjoint union is over the subset of all trivalent ribbon graphs (with labelled boundaries) satisfying the topological stability condition $2 - 2g - N_0(T) < 0$, and which are dual to generalized triangulations. It follows, (see [8] theorems 3.3, 3.4, and 3.5), that the set $K_1 RP_{g,N_0}^{met}$ is locally modelled on a stratified space constructed from the components (rational orbicells) $\mathbb{R}_+^{|e(\Gamma)|}/Aut_\partial(P_l)$ by means of a (Whitehead) expansion and collapse procedure for ribbon graphs, which amounts to collapsing edges and coalescing vertices, (the Whitehead move in $|P_{T_l}| \to M$ is the dual of the familiar flip move for triangulations). Explicitly, if $l(t) = tl$ is the length of an edge $\rho^1(j)$ of a ribbon graph $\Gamma_{l(t)} \in K_1 RP_{g,N_0}^{met}$, then, as $t \to 0$, we get the metric ribbon graph $\widehat{\Gamma}$ which is obtained from $\Gamma_{l(t)}$ by collapsing the edge $\rho^1(j)$. By exploiting such construction, we can extend the space $K_1 RP_{g,N_0}^{met}$ to a suitable closure $\overline{K_1 RP}_{g,N_0}^{met}$ [10], (this natural topology on $K_1 RP_{g,N_0}^{met}$ shows that, at least in two-dimensional quantum gravity, the set of Regge triangulations with *fixed connectivity* does not explore the full configurational space of the theory). The open cells of $K_1 RP_{g,N_0}^{met}$, being associated with trivalent graphs, have dimension provided by the number $N_1(T)$ of edges of $|P_{T_l}| \to M$. From the Euler relation $N_0(T) - N_1(T) + N_2(T) = 2 - 2g$, and the constraint $2N_1(T) = 3N_2(T)$ associated with the trivalency, we get

$$\dim \left[ K_1 RP_{g,N_0}^{met} \right] = N_1(T) = 3N_0(T) + 6g - 6. \tag{22}$$

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There is a natural projection

$$p : K_1 RP^{met}_{g,N_0} \longrightarrow \mathbb{R}^{N_0(T)}_+ \qquad (23)$$

$$\Gamma \longmapsto p(\Gamma) = (l_1, ..., l_{N_0(T)}),$$

where $(l_1, ..., l_{N_0(T)})$ denote the perimeters of the polygonal 2-cells $\{\rho^2(j)\}$ of $|P_{T_l}| \to M$. With respect to the topology on the space of metric ribbon graphs, the orbifold $K_1 RP^{met}_{g,N_0}$ endowed with such a projection acquires the structure of a cellular bundle. For a given sequence $\{l(\partial(\rho^2(k)))\}$, the fiber

$$p^{-1}(\{l(\partial(\rho^2(k)))\}) = \{|P_{T_l}| \to M \in K_1 RP^{met}_{g,N_0} : \{l_k\} = \{l(\partial(\rho^2(k)))\}\} \quad (24)$$

is the set of all generalized conical Regge polytopes with the given set of perimeters. If we take into account the $N_0(T)$ constraints associated with the perimeters assignments, it follows that the fibers $p^{-1}(\{l(\partial(\rho^2(k)))\})$ have dimension provided by

$$\dim \left[ p^{-1}(\{l(\partial(\rho^2(k)))\}) \right] = 2N_0(T) + 6g - 6, \qquad (25)$$

which exactly corresponds to the real dimension of the moduli space $\mathfrak{M}_{g,N_0}$ of genus $g$ Riemann surfaces $((M; N_0), \mathcal{C})$ with $N_0$ punctures.

## 2.6. Orbifold labelling and dynamical triangulations.

Let us denote by

$$\Omega_{T_a} \doteq \frac{\mathbb{R}^{|e(\Gamma)|}_+}{Aut_\partial(P_{T_a})} \qquad (26)$$

the rational cell associated with the 1-skeleton of the conical polytope $|P_{T_a}| \to M$ dual to a dynamical triangulation $|T_{l=a}| \to M$. The orbicell (26) contains the ribbon graph associated with $|P_{T_a}| \to M$ and all (trivalent) metric ribbon graphs $|P_{T_L}| \to M$ with the same combinatorial structure of $|P_{T_a}| \to M$ but with all possible length assignments $\{l(\rho^1(h))\}_1^{N_1(T)}$ associated with the corresponding set of edges $\{\rho^1(h)\}_1^{N_1(T)}$. The orbicell $\Omega_{T_a}$ is naturally identified with the convex polytope (of dimension $(2N_0(T) + 6g - 6)$) in $\mathbb{R}^{N_1(T)}_+$ defined by

$$\left\{ \{l(\rho^1(j))\} \in \mathbb{R}^{N_1(T)}_+ : \sum_{j=1}^{q(k)} A^j_{(k)}(T_a) L(\rho^1(j)) = \frac{\sqrt{3}}{3} aq(k), \ k = 1, ..., N_0 \right\},$$

$$(27)$$

where $A^j_{(k)}(T_a)$ is a $(0,1)$ indicator matrix, depending on the given dynamical triangulation $|T_{l=a}| \to M$, with $A^j_{(k)}(T_a) = 1$ if the edge $\rho^1(j)$ belongs to $\partial(\rho^2(k))$, and 0 otherwise, and $\frac{\sqrt{3}}{3} aq(k)$ is the perimeter length $l(\partial(\rho^2(k)))$ in terms of the corresponding curvature assignment $q(k)$. Note that $|P_{T_a}| \to M$ appears as the barycenter of such a polytope.

Since the cell decomposition (21) of the space of trivalent metric ribbon graphs $K_1 RP^{met}_{g,N_0}$ depends only on the combinatorial type of the ribbon graph,

we can use the equilateral polytopes $|P_{T_a}| \to M$, dual to dynamical triangulations, as the set over which the disjoint union in (21) runs. Thus we can write

$$K_1 R P_{g,N_0}^{met} = \bigsqcup_{\mathcal{DT}(N_0)} \Omega_{T_a}, \qquad (28)$$

where

$$\mathcal{DT}_g(N_0) \doteq \left\{ |T_{l=a}| \to M \; : (\sigma^0(k)) \; k = 1, ..., N_0(T) \right\} \qquad (29)$$

denote the set of distinct generalized dynamically triangulated surfaces of genus $g$, with a given set of $N_0(T)$ ordered labelled vertices.

Note that [11], even if the set $\mathcal{DT}_g(N_0)$ can be considered (through barycentrical dualization) a well-defined subset of $K_1 R P_{g,N_0}^{met}$, it is not an orbifold over $\mathbb{N}$. For this latter reason, the analysis of the metric stuctures over (generalized) polytopes requires the use of the full orbicells $\Omega_{T_a}$ and we cannot limit our discussion to equilateral polytopes.

# 3  Some properties of the moduli space.

At this point, it is worthwhile to discuss a few general aspects of moduli space theory which come up in our geometrical analysis of triangulated surfaces. We will do no more than summarize a few basic results and discuss in some detail only the aspects of the theory which are of particular relevance to us. We start by recalling that the moduli space $\mathfrak{M}_{g,N_0}$ of genus $g$ Riemann surfaces $((M; N_0), \mathcal{C})$ with $N_0$ punctures is a dense open subset of a natural compactification (Knudsen-Deligne-Mumford ) in a connected, compact orbifold of complex dimension $3g - 3 + N_0$ denoted by $\overline{\mathfrak{M}}_{g,N_0}$. This latter is, by definition, the moduli space of stable $N_0$-pointed curves of genus $g$, where a stable curve is a compact Riemann surface with at most ordinary double points such that its parts are hyperbolic. The closure $\partial \mathfrak{M}_{g,N_0}$ of $\mathfrak{M}_{g,N_0}$ in $\overline{\mathfrak{M}}_{g,N_0}$ consists of stable curves with double points, and gives rise to a stratification decomposing $\overline{\mathfrak{M}}_{g,N_0}$ into subvarieties. By definition, a stratum of codimension $k$ is the component of $\overline{\mathfrak{M}}_{g,N_0}$ parametrizing stable curves (of fixed topological type) with $k$ double points.

### 3.1. Surface bundles and their canonical sections.

A basic observation in moduli space theory is the fact that any point $p$ on a stable curve $((M; N_0), \mathcal{C}) \in \overline{\mathfrak{M}}_{g,N_0}$ defines a natural mapping

$$((M; N_0), \mathcal{C}) \longrightarrow \overline{\mathfrak{M}}_{g,N_0+1} \qquad (30)$$

that determines a stable curve $((M; N_0 + 1), \mathcal{C}') \in \overline{\mathfrak{M}}_{g,N_0+1}$. Explicitly, as long as the point $p$ is disjoint from the puncture set $\{p_k\}_{k=1}^{N_0}$ one simply defines $((M; N_0 + 1), \mathcal{C}')$ to be $((M; N_0, \{p\}), \mathcal{C})$. If the point $p = p_h$ for some $p_h \in \{p_k\}_{k=1}^{N_0}$, then: *i)* for any $1 \le i \le N_0$, with $i \ne h$, identify $p_i' \in ((M; N_0 + 1), \mathcal{C}')$

with the corresponding $p_i$; *ii)* take a three punctured sphere $\mathbb{CP}^1_{(0,1,\infty)}$, label with a sub-index $h$ one of its punctures $(0,1,\infty)$, say $\infty_h$, and attach it to the given $p_h \in [((M;N_0),\mathcal{C})]$; *iii)* relabel the remaining two punctures $(0,1) \in \mathbb{CP}^1_{(0,1,\infty)}$ as $p'_h$ and $p'_{N_0+1}$. In this way, we get a genus $g$ noded surface

$$s_h\left[((M;N_0),\mathcal{C})\right] = [((M;N_0+1),\mathcal{C}')] \doteq ((M;N_0),\mathcal{C})_h \cup \mathbb{CP}^1_{(0,1,\infty_h)} \qquad (31)$$

with a rational tail and with a double point corresponding to the original puncture $p_h$. Finally if $p$ happens to coincide with a node, then $[((M;N_0+1),\mathcal{C}')]$ results from setting $p'_j \doteq p_j$ for any $1 \leq i \leq N_0$ and by: *i)* normalizing $[((M;N_0),\mathcal{C})]$ at the node ( *i.e.*, by separating the branches of $[((M;N_0),\mathcal{C})]$ at $p$); *ii)* inserting a copy of $\mathbb{CP}^1_{(0,1,\infty)}$ with $\{0,\infty\}$ identified with the preimage of $p$ and with $p'_{N_0+1} \doteq 1 \in \mathbb{CP}^1_{(0,1,\infty)}$. Conversely, let

$$\pi : \overline{\mathfrak{M}}_{g,N_0+1} \longrightarrow \overline{\mathfrak{M}}_{g,N_0} \qquad (32)$$

$$[((M;N_0+1),\mathcal{C})] \vdash \begin{array}{c} forget \\ \& \\ collapse \end{array} \longrightarrow [((M;N_0),\mathcal{C}')]$$

the projection which forgets the $(N_0+1)^{st}$ puncture and collapse to a point any irreducible unstable component of the resulting curve. The fiber of $\pi$ over $((M;N_0),\mathcal{C})$ is parametrized by the map (30), and if $((M;N_0),\mathcal{C})$ has a trivial automorphism group $Aut[((M;N_0),\mathcal{C})]$ then $\pi^{-1}((M;N_0),\mathcal{C})$ is by definition the surface $((M;N_0),\mathcal{C})$, otherwise it is identified with the quotient $((M;N_0),\mathcal{C})/Aut[((M;N_0),\mathcal{C})]$. Thus, under the action of $\pi$, we can consider $\overline{\mathfrak{M}}_{g,N_0+1}$ as a family (in the orbifold sense) of Riemann surfaces over $\overline{\mathfrak{M}}_{g,N_0}$ and we can identify $\overline{\mathfrak{M}}_{g,N_0+1}$ with the universal curve $\overline{\mathcal{C}}_{g,N_0}$,

$$\pi : \overline{\mathcal{C}}_{g,N_0} \longrightarrow \overline{\mathfrak{M}}_{g,N_0} . \qquad (33)$$

Note that, by construction, $\overline{\mathcal{C}}_{g,N_0}$ (but for our purposes is more profitable to think in terms of $\overline{\mathfrak{M}}_{g,N_0+1}$) comes endowed with the $N_0$ natural sections $s_1,...,s_{N_0}$

$$s_h : \overline{\mathfrak{M}}_{g,N_0} \longrightarrow \overline{\mathcal{C}}_{g,N_0} \qquad (34)$$

$$[((M;N_0),\mathcal{C})] \longmapsto s_h\left[((M;N_0),\mathcal{C})\right] \doteq ((M;N_0),\mathcal{C})_h \cup \mathbb{CP}^1_{(0,1,\infty_h)},$$

defined by (31).

## 3.2. The relative dualizing sheaf.

The images of the sections $s_i$ characterize a divisor $\{D_i\}_{i=1}^{N_0}$ in $\overline{\mathcal{C}}_{g,N_0}$ which has a great geometric interest both in quantum gravity (where it is associated with the generation of MinBUs: Minimal Bottleneck Universes, the configurations characterizing the susceptibility exponent of 2D gravity), and in discussing the topology of $\overline{\mathfrak{M}}_{g,N_0}$. In both cases, such a study exploits the properties of the tautological classes over $\overline{\mathcal{C}}_{g,N_0}$ generated by the sections $\{s_i\}_{i=1}^{N_0}$ and by the

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corresponding divisors $\{D_i\}_{i=1}^{N_0}$. To define such classes, recall that the cotangent bundle (in the orbifold sense) to the fibers of of the universal curve $\pi : \overline{\mathcal{C}}_{g,N_0} \longrightarrow \overline{\mathfrak{M}}_{g,N_0}$ gives rise to a holomorphic line bundle $\omega_{g,N_0} \doteq \omega_{\overline{\mathcal{C}}_{g,N_0}/\overline{\mathfrak{M}}_{g,N_0}}$ over $\overline{\mathcal{C}}_{g,N_0}$ (the relative dualizing sheaf of $\pi : \overline{\mathcal{C}}_{g,N_0} \longrightarrow \overline{\mathfrak{M}}_{g,N_0}$), this is essentially the sheaf of 1-forms with a natural polar behavior along the possible nodes of the Riemann surface describing the fiber of $\pi$. A more explicit characterization, however, will be needed later on, so we briefly pause to describe it here. In particular, we will be interested on the behavior of the relative dualizing sheaf $\omega_{g,N_0}$ restricted to the generic divisor $D_h$ generated by the section $s_h$. To this end, let $z_1(h)$ and $z_2(\infty_h)$ denote local coordinates defined in the disks $\Delta_{p_h} \doteq \{|z_1(h)| < 1\}$ and $\Delta_{\infty_h} \doteq \{|z_2(\infty_h)| < 1\}$ respectively centered around the punctures $p_h \in ((M; N_0), \mathcal{C})$, and $\infty \in \mathbb{CP}^1_{(0,1,\infty)}$. Let $\Delta_{t_h} = \{t_h \in \mathbb{C} : |t_h| < 1\}$. Consider the analytic family $s_h(t_h)$ of surfaces of genus $g$ defined over $\Delta_{t_h}$ and obtained by removing the disks $|z_1(h)| < |t_h|$ and $|z_2(\infty_h)| < |t_h|$ from $((M; N_0), \mathcal{C})$ and $\mathbb{CP}^1_{(0,1,\infty)}$ and gluing the resulting surfaces through the annulus $\{(z_1(h), z_2(\infty_h))| \; z_1(h)z_2(\infty_h) = t_h, \; t_h \in \Delta_{t_h}\}$ by identifying the points of coordinate $z_1(h)$ with the points of coordinates $z_2(\infty_h) = t_h/z_1(h)$. The family $s_h(t_h) \to \Delta_{t_h}$ opens the node $z_1(h)z_2(\infty_h) = 0$ of the section $s_h|_{((M;N_0),\mathcal{C})}$. Note that in such a way we can independently and holomorphically open the distinct nodes of the various sections $\{s_k\}_{k=1}^{N_0}$. More generally, while opening the node we can also vary the complex structure of $((M; N_0), \mathcal{C})$ by introducing local complex coordinates $(\tau_\alpha)_{\alpha=1}^{3g-3+N_0}$ for $\mathfrak{M}_{g,N_0}$ around $((M; N_0), \mathcal{C})$. If

$$s_h(\tau_\alpha, t_h) \to \mathfrak{M}_{g,N_0} \times \Delta_{t_h} \qquad (35)$$

denotes the family of surfaces opening of the node , then in the corresponding coordinates $(\tau_\alpha, t_h)$ the divisor $D_h$, image of the section $s_h$, is locally defined by the equation $t_h = 0$. Similarly, the divisor $D \doteq \sum_{h=1}^{N_0} D_h$ is characterized by the locus of equation $\prod_{h=1}^{N_0} t_h = 0$.

The elements of the dualizing sheaf $\omega_{g,N_0}|_{s_h(t_h)} \doteq \omega_{g,N_0}(D_h)$ are differential forms $u(h) = u_1 dz_1(h) + u_2 dz_2(\infty_h)$ such that $u(h) \wedge dt_h = f dz_1(h) \wedge dz_2(\infty_h)$, where $f$ is a holomorphic function of $z_1(h)$ and $z_2(\infty_h)$. By differentiating $z_1(h)z_2(\infty_h) = t_h$, one gets $f = u_1 z_1(h) - u_2 z_2(\infty_h)$ which is the defining relation for the forms in $\omega_{g,N_0}(D_h)$. In particular, by choosing $u_1 = f/2z_1(h)$, and $u_2 = f/2z_2(\infty_h)$ we get the local isomorphism between the sheaf of holomorphic functions $\mathcal{O}_{s_h(t_h)}$ over $s_h(t_h)$ and $\omega_{g,N_0}(D_h)$

$$f \longmapsto u(h) = f \left( \frac{1}{2} \frac{dz_1(h)}{z_1(h)} - \frac{1}{2} \frac{dz_2(\infty_h)}{z_2(\infty_h)} \right). \qquad (36)$$

If we set $f = f_0 + f_1(z_1(h)) + f_2(z_2(\infty_h))$, where $f_0$ is a constant and $f_1(0) = 0 = f_2(0)$, then on the noded surface $s_h$ , $(t_h = 0)$, we get from the relation

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$$z_2(\infty_h)dz_1(h) + z_1(h)dz_2(\infty_h) = 0,$$

$$
\begin{aligned}
u_h|_{z_2(\infty_h)=0} &= \frac{f_0 + f_1(z_1(h))}{z_1(h)}dz_1(h), \\
u_h|_{z_1(h)=0} &= -\frac{f_0 + f_2(z_2(\infty_h))}{z_2(\infty_h)}dz_2(\infty_h),
\end{aligned}
\tag{37}
$$

on the two branches $\Delta_{p_h} \cap ((M;N_0),\mathcal{C})$ and $\Delta_{\infty_h} \cap \mathbb{CP}^1_{(0,1,\infty)}$ of the node where $z_1(h)$ and $z_2(\infty_h)$ are a local coordinate (*i.e.* $z_2(\infty_h) = 0$ and $z_1(h) = 0$, respectively). Thus, near the node of $s_h$, $\omega_{g,N_0}(D_h)$ is generated by $\frac{dz_1(h)}{z_1(h)}$ and $\frac{dz_2(\infty_h)}{z_2(\infty_h)}$ subjected to the relation $\frac{dz_1(h)}{z_1(h)} + \frac{dz_2(\infty_h)}{z_2(\infty_h)} = 0$. Stated differently, a section of the sheaf $\omega_{g,N_0}(D_h)$ pulled back to the smooth normalization $((M;N_0),\mathcal{C})_{p_h} \bigsqcup \mathbb{CP}^1_{(0,1,\infty_h)}$ of $s_h$ can be identified with a meromorphic 1-form with at most simple poles at the punctures $p_h$ and $\infty_h$ which are identified under the normalization map, and with opposite residues at such punctures. By extending such a construction to all $N_0$ sections $\{s_h\}_{h=1}^{N_0}$, we can define the line bundle

$$\omega_{g,N_0}(D) \doteq \omega_{g,N_0}\left(\sum_{i=1}^{N_0} D_i\right) \longrightarrow \overline{\mathcal{C}}_{g,N_0} \tag{38}$$

as $\omega_{g,N_0}$ twisted by the divisor $D \doteq \sum_{h=1}^{N_0} D_h$, *viz.* the line bundle locally generated by the differentials $\frac{dz_1(h)}{z_1(h)}$ for $z_1(h) \neq 0$ and $-\frac{dz_2(\infty_h)}{z_2(\infty_h)}$ for $z_2(\infty_h) \neq 0$, with $z_1(h)z_2(\infty_h) = 0$, and $h = 1,...,N_0$. As above, $\{z_1(h)\}_{h=1}^{N_0}$ are local variables at the marked points $\{p_h\}_{i=1}^{N_0} \in ((M;N_0),\mathcal{C})$, whereas $z_2(\infty_h)$ is the corresponding variable in the three punctured sphere $\mathbb{CP}^1_{(0,1,\infty)}$.

### 3.3. Meromorphic quadratic differentials and their Weil-Petersson norm.

We can associate with the family of surfaces $s_h(\tau_\alpha, t_h)$, opening the node $t_h = 0$ of the section $s_h(\tau_\alpha)$, a meromorphic quadratic differential which, in terms of the local covering for $\mathfrak{M}_{g,N_0}$ defined by (35), is given by [12]

$$\Phi(h;\tau_\alpha,t_h) \doteq \begin{cases} -\frac{t_h}{\pi}\frac{dz_1(h)\otimes dz_1(h)}{z_1(h)^2}, & \text{in} \quad \Delta_{p_h} \cap ((M;N_0),\mathcal{C}) \\ -\frac{t_h}{\pi}\frac{dz_2(\infty_h)\otimes dz_2(\infty_h)}{z_2(\infty_h)^2}, & \text{in} \quad \Delta_{\infty_h} \cap \mathbb{CP}^1_{(0,1,\infty)}. \end{cases} \tag{39}$$

The quadratic differential $\Phi(h;\tau_\alpha,t_h)$ is an element of the (relative) 2-canonical bundle $\omega_{g,N_0}(D_h)^{\otimes 2}$ originating from the basis $\frac{dz_1(h)}{z_1(h)}$ of $\omega_{g,N_0}(D_h)$, so, it can be thought of as a cotangent vector to $\mathfrak{M}_{g,N_0}$, dual to the holomorphic vector field $\frac{\partial}{\partial t_h}$. In any annulus $\{|t_h| < |z_1(h)| < 1\}$ parametrizing the opening of the node family $s_h(\tau_\alpha, t_h)$, we can evaluate the Weil-Petersson norm [12] of the quadratic differential $\Phi(h;\tau_\alpha,t_h)$ according to

$$\|\Phi(h;\tau_\alpha,t_h)\|_{W-P} = \int_{\{|t_h|<|z_1(h)|<1\}} \frac{|\Phi(h;\tau_\alpha,t_h)|^2}{g_{hyp}}, \tag{40}$$

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where $g_{hyp}$ denotes the hyperbolic metric

$$g_{hyp} \doteq \left( \frac{\pi^2}{\ln^2 |t_h|} \right) \frac{dz_1(h)d\overline{z_1(h)}}{|z_1(h)|^2 \sin^2 \left( \pi \frac{\ln |z_1(h)|}{\ln |t_h|} \right)} \tag{41}$$

on $\{|t_h| < |z_1(h)| < 1\}$. A direct computation provides [12]

$$\|\Phi(h; \tau_\alpha, t_h)\|_{W-P} = |t_h|^2 \left( \frac{1}{\pi} \ln \frac{1}{|t_h|} \right)^3. \tag{42}$$

The corresponding Weil–Petersson metric $ds_{W-P}^2(h)$ and the associated Kähler form $\omega_{W-P}(h)$ are respectively provided by

$$ds_{W-P}^2(h) = \frac{2\pi^3 |dt_h|^2}{|t_h|^2 \left( \ln \frac{1}{|t_h|} \right)^3}, \tag{43}$$

and

$$\omega_{W-P}(h) = \sqrt{-1} \frac{2\pi^3}{|t_h|^2 \left( \ln \frac{1}{|t_h|} \right)^3} dt_h \wedge d\overline{t_h}. \tag{44}$$

Observe here a similarity between the above definitions and properties of the sections $\{s_h\}$ and the construction of minimal bottleneck universes in simplicial quantum gravity. This is not a mere coincidence, and it will turn out that such a similarity has a far reaching role in discussing the complex analytic geometry of Regge polytopes and its connection with 2D simplicial quantum gravity.

### 3.4. The ribbon graph parametrization of the moduli space.

The complex analytic geometry of the space of conical Regge polytopes which we will discuss in the next section generalizes the well-known bijection (a homeomorphism of orbifolds) between the space of metric ribbon graphs $K_1 RP_{g,N_0}^{met}$ (which forgets the conical geometry) and the moduli space $\mathfrak{M}_{g,N_0}$ of genus $g$ Riemann surfaces $((M; N_0), \mathcal{C})$ with $N_0(T)$ punctures [8], [10]. This bijection results in a local parametrization of $\mathfrak{M}_{g,N_0}$ defined by

$$h : K_1 RP_{g,N_0}^{met} \rightarrow \mathfrak{M}_{g,N_0} \times R_+^N \tag{45}$$
$$\Gamma \longmapsto [((M; N_0), \mathcal{C}), l_i]$$

where $(l_1, ..., l_{N_0})$ is an ordered n-tuple of positive real numbers and $\Gamma$ is a metric ribbon graph with $N_0(T)$ labelled boundary lengths $\{l_i\}$. If $\overline{K_1 RP}_{g,N_0}^{met}$ is the closure of $K_1 RP_{g,N_0}^{met}$, then the bijection $h$ extends to $\overline{K_1 RP}_{g,N_0}^{met} \rightarrow \overline{\mathfrak{M}}_{g,N_0} \times R_+^{N_0}$ in such a way that a ribbon graph $\Gamma \in \overline{RGP}_{g,N_0}^{met}$ is mapped in two (stable) surfaces $M_1$ and $M_2$ with $N_0(T)$ punctures if and only if there exists an homeomorphism between $M_1$ and $M_2$ preserving the (labelling of the) punctures, and is holomorphic on each irreducible component containing one of the punctures.
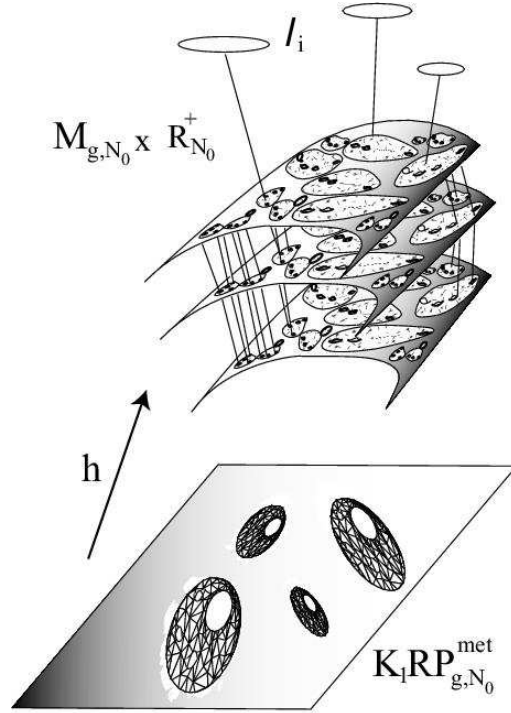
Figure 6: The map $h$ associates to each ribbon graph an element of the standard moduli space $\mathfrak{M}_{g,N_0} \times \mathfrak{R}^+_{N_0}$.

According to Kontsevich [13], corresponding to each marked polygonal 2-cells $\{\rho^2(k)\}$ of $|P_{T_l}| \to M$ there is a further (combinatorial) bundle map

$$\mathcal{CL}_k \to K_1 RP^{met}_{g,N_0} \tag{46}$$

whose fiber over $(\Gamma, \rho^2(1), ..., \rho^2(N_0))$ is provided by the boundary cycle $\partial \rho^2(k)$, (recall that each boundary $\partial \rho^2(k)$ comes with a positive orientation).
To any such cycle one associates [17], [10] the corresponding perimeter map $l(\partial(\rho^2(k))) = \sum l(\rho^1(h_\alpha))$ which then appears as defining a natural connection on $\mathcal{CL}_k$. The piecewise smooth 2-form defining the curvature of such a connection,

$$\omega_k(\Gamma) = \sum_{1 \le h_\alpha < h_\beta \le q(k)-1} d\left(\frac{l(\rho^1(h_\alpha))}{l(\partial\rho^2(k))}\right) \wedge d\left(\frac{l(\rho^1(h_\beta))}{l(\partial\rho^2(k))}\right), \tag{47}$$

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is invariant under rescaling and cyclic permutations of the $l(\rho^1(h_\mu))$, and is a combinatorial representative of the Witten class of the line bundle $\mathcal{L}_k$.

It is important to stress that even if ribbon graphs can be thought of as arising from Regge polytopes (with variable connectivity), (45) only involves the ribbon graph structure and the theory can be (and actually is) developed with no reference at all to a particular underlying triangulation. In such a connection, the role of dynamical triangulations has been slightly overemphasized, they simply provide a convenient way of labelling the different combinatorial strata of the mapping (45), but, by themselves they do not define a combinatorial parametrization of $\overline{\mathfrak{M}}_{g,N_0}$ for any finite $N_0$. However, it is very useful, at least for the purposes of quantum gravity, to remember the possible genesis of a ribbon graph from an underlying triangulation and be able to exploit the further information coming from the associated conical geometry. Such an information cannot be recovered from the ribbon graph itself (with the notable exception of equilateral ribbon graphs, which can be associated with dynamical triangulations), and must be suitably codified by adding to the boundary lengths $\{l_i\}$ of the graph a further decoration. This can be easily done by explicitly connecting Regge polytopes to punctures Riemann surfaces.

# 4  Regge polytopes and punctured Riemann surfaces.

As suggested by (1), the polyhedral metric associated with the vertices $\{\sigma^0(i)\}$ of a (generalized) Regge triangulation $|T_l| \to M$, can be conveniently described in terms of complex function theory. We can associate with $|P_{T_l}| \to M$ a complex structure $((M; N_0), \mathcal{C})$ (a punctured Riemann surface) which is, in a well-defined sense, dual to the structure (1) generated by $|T_l| \to M$. Let $\rho^2(k)$ be the generic two-cell $\in |P_{T_l}| \to M$ barycentrically dual to the vertex $\sigma^0(k) \in |T_l| \to M$. To the generic edge $\rho^1(h)$ of $\rho^2(k)$ we associate a complex uniformizing coordinate $z(h)$ defined in the strip

$$U_{\rho^1(h)} \doteq \{z(h) \in \mathbb{C}| 0 < \mathrm{Re}\, z(h) < l(\rho^1(h))\}, \tag{48}$$

$l(\rho^1(h))$ being the length of the edge considered. The uniformizing coordinate $w(j)$, corresponding to the generic 3-valent vertex $\rho^0(j) \in \rho^2(k)$, is defined in the open set

$$U_{\rho^0(j)} \doteq \{w(j) \in \mathbb{C}| \, |w(j)| < \delta, \, w(j)[\rho^0(j)] = 0\}, \tag{49}$$

where $\delta > 0$ is a suitably small constant. Finally, the two-cell $\rho^2(k)$ is uniformized in the unit disk

$$U_{\rho^2(k)} \doteq \{\zeta(k) \in \mathbb{C}| \, |\zeta(k)| < 1, \, \zeta(k)[\sigma^0(k)] = 0\}, \tag{50}$$

where $\sigma^0(k)$ is the vertex $\in |T_l| \to M$ corresponding to the given two-cell.

The various uniformizations $\{w(j), U_{\rho^0(j)}\}_{j=1}^{N_2(T)}$, $\{z(h), U_{\rho^1(h)}\}_{h=1}^{N_1(T)}$, and $\{\zeta(k), U_{\rho^2(k)}\}_{k=1}^{N_0(T)}$ can be coherently glued together by noting that to each edge $\rho^1(h) \in \rho^2(k)$ we can associate the standard quadratic differential on $U_{\rho^1(h)}$ given by

$$\phi(h)|_{\rho^1(h)} = dz(h) \otimes dz(h). \tag{51}$$

Such $\phi(h)|_{\rho^1(h)}$ can be extended to the remaining local uniformizations $U_{\rho^0(j)}$, and $U_{\rho^2(k)}$, by exploiting a classic result in Riemann surface theory according to which a quadratic differential $\phi$ has a finite number of zeros $n_{zeros}(\phi)$ with orders $k_i$ and a finite number of poles $n_{poles}(\phi)$ of order $s_i$ such that

$$\sum_{i=1}^{n_{zero}(\phi)} k_i - \sum_{i=1}^{n_{pole}(\phi)} s_i = 4g - 4. \tag{52}$$

In our case we must have $n_{zeros}(\phi) = N_2(T)$ with $k_i = 1$, (corresponding to the fact that the 1-skeleton of $|P_l| \to M$ is a trivalent graph), and $n_{poles}(\phi) = N_0(T)$ with $s_i = s \; \forall i$, for a suitable positive integer $s$. According to such remarks (52) reduces to

$$N_2(T) - sN_0(T) = 4g - 4. \tag{53}$$

From the Euler relation $N_0(T) - N_1(T) + N_2(T) = 2 - 2g$, and $2N_1(T) = 3N_2(T)$ we get $N_2(T) - 2N_0(T) = 4g - 4$. This is consistent with (53) if and only if $s = 2$. Thus the extension $\phi$ of $\phi(h)|_{\rho^1(h)}$ along the 1-skeleton of $|P_l| \to M$ must have $N_2(T)$ zeros of order 1 corresponding to the trivalent vertices $\{\rho^0(j)\}$ of $|P_l| \to M$ and $N_0(T)$ quadratic poles corresponding to the polygonal cells $\{\rho^2(k)\}$ of perimeter lengths $\{l(\partial(\rho^2(k)))\}$.

Around a zero of order one and a pole of order two, every (Jenkins-Strebel) quadratic differential $\phi$ has a canonical local structure which (along with (51)) is given by [8], [16] ,

$$(|P_{T_l}| \to M) \to \phi \doteq \begin{cases} \phi(h)|_{\rho^1(h)} = dz(h) \otimes dz(h), \\ \phi(j)|_{\rho^0(j)} = \frac{9}{4} w(j) dw(j) \otimes dw(j), \\ \phi(k)|_{\rho^2(k)} = -\frac{[l(\partial(\rho^2(k)))]^2}{4\pi^2 \zeta^2(k)} d\zeta(k) \otimes d\zeta(k), \end{cases} \tag{54}$$

where $\{\rho^0(j), \rho^1(h), \rho^2(k)\}$ runs over the set of vertices, edges, and 2-cells of $|P_l| \to M$. Since $\phi(h)|_{\rho^1(h)}$, $\phi(j)|_{\rho^0(j)}$, and $\phi(k)|_{\rho^2(k)}$ must be identified on the non-empty pairwise intersections $U_{\rho^0(j)} \cap U_{\rho^1(h)}$, $U_{\rho^1(h)} \cap U_{\rho^2(k)}$ we can associate to the polytope $|P_{T_l}| \to M$ a complex structure $((M; N_0), \mathcal{C})$ by co-herently gluing, along the pattern associated with the ribbon graph $\Gamma$, the local uniformizations $\{U_{\rho^0(j)}\}_{j=1}^{N_2(T)}$, $\{U_{\rho^1(h)}\}_{h=1}^{N_1(T)}$, and $\{U_{\rho^2(k)}\}_{k=1}^{N_0(T)}$. Explicitly, let $\{U_{\rho^1(j_\alpha)}\}$, $\alpha = 1, 2, 3$ be the three generic open strips associated with the three cyclically oriented edges $\{\rho^1(j_\alpha)\}$ incident on the generic vertex $\rho^0(j)$. Then the uniformizing coordinates $\{z(j_\alpha)\}$ are related to $w(j)$ by the transition functions

$$w(j) = e^{2\pi i \frac{\alpha-1}{3}} z(j_\alpha)^{\frac{2}{3}}, \qquad \alpha = 1, 2, 3. \tag{55}$$
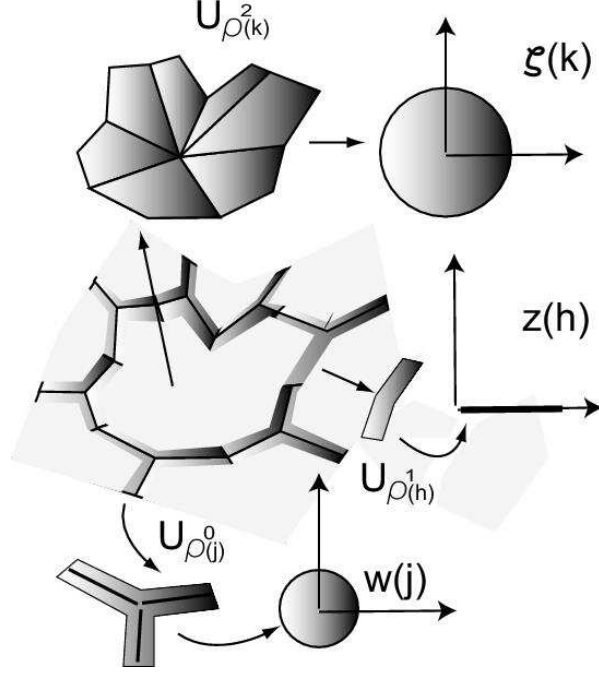
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Figure 7: The local presentation of uniformizing coordinates.

Note that in such uniformization the vertices $\{ \rho^0(j) \}$ do not support conical singularities since each strip $U_{\rho^1(j_\alpha)}$ is mapped by (55) into a wedge of angular opening $\frac{2\pi}{3}$. This is consistent with the definition of $|P_{T_l}| \to M$ according to which the vertices $\{ \rho^0(j) \} \in |P_{T_l}| \to M$ are the barycenters of the flat $\{ \sigma^2(j) \} \in |T_l| \to M$. Similarly, if $\{ U_{\rho^1(k_\beta)} \}$, $\beta = 1, 2, ..., q(k)$ are the open strips associated with the $q(k)$ (oriented) edges $\{ \rho^1(k_\beta) \}$ boundary of the generic polygonal cell $\rho^2(k)$, then the transition functions between the corresponding uniformizing coordinate $\zeta(k)$ and the $\{ z(k_\beta) \}$ are given by [8]

$$\zeta(k) = \exp\left( \frac{2\pi i}{l(\partial(\rho^2(k)))} \left( \sum_{\beta=1}^{\nu-1} l(\rho^1(k_\beta)) + z(k_\nu) \right) \right), \quad \nu = 1, ..., q(k), \quad (56)$$

with $\sum_{\beta=1}^{\nu-1} \cdot \doteq 0$, for $\nu = 1$.

## 4.1. A Parametrization of the conical geometry.

As for the metrical properties of the complex structure $((M; N_0), \mathcal{C})$ associated with $|P_{T_l}| \to M$ note that for any closed curve $c : \mathbb{S}^1 \to U_{\rho^2(k)}$, homotopic to

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the boundary of $\overline{U}_{\rho^2(k)}$, we get

$$\oint_c \sqrt{\phi(k)_{\rho^2(k)}} = l(\partial(\rho^2(k))). \tag{57}$$

In general, let

$$|\phi(k)_{\rho^2(k)}| = \frac{\left[l(\partial(\rho^2(k)))\right]^2}{4\pi^2|\zeta(k)|^2}|d\zeta(k)|^2, \tag{58}$$

denote the usual cylindrical metric canonically associated with a quadratic differential with a second order pole. If we define $\Delta_k^\varrho \doteq \{\zeta(k) \in \mathbb{C}|\ \varrho \le |\zeta(k)| \le 1\}$, then in terms of the area element associated with the flat metric $|\phi(k)_{\rho^2(k)}|$, (i.e., the absolute value of $\phi(k)$), we get

$$\int_{\Delta_k^\varrho} \frac{i}{2} \frac{\left[l(\partial(\rho^2(k)))\right]^2}{4\pi^2|\zeta(k)|^2}d\zeta(k) \wedge d\overline{\zeta}(k) = \frac{\left[l(\partial(\rho^2(k)))\right]^2}{2\pi}\ln\left(\frac{1}{\varrho}\right), \tag{59}$$

which, as $\varrho \to 0^+$, diverges logarithmically. Thus, as already recalled (section 2.3) the punctured disk $\Delta_k^*$, endowed with the flat metric $|\phi(k)_{\rho^2(k)}|$, is isometric to a flat semi-infinite cylinder. It is perhaps worthwhile stressing that even if this latter geometry is perfectly consistent with the metric ribbon graph structure, it is not really the natural metric to use if we wish to explicitly keep track of the conical Regge polytope (and the associated triangulation) which generate the given ribbon graph. For instance, in the Kontsevich-Witten model, the ribbon graph structure and the associated geometry $(\Delta_k^*, |\phi(k)_{\rho^2(k)}|)$ of the cells $U_{\rho^2(k)}$ is sufficient to combinatorially describe intersection theory on moduli space. However, a full-fledged study of 2D simplicial quantum gravity (e.g., of Liouville theory) requires a study of the full Regge geometry $(\Delta_k^*, ds_{(k)}^2)$.

We can keep track of the conical geometry of the polygonal cell $\rho^2(k) \in |P_{T_l}| \to M$ by noticing that given a normalized deficit angle $\frac{\varepsilon(k)}{2\pi} \doteq 1 - \frac{\theta(k)}{2\pi}$, the conical geometry (1) and the cylindrical geometry (58) on the punctured disk $\Delta_k^* \subset U_{\rho^2(k)}$

$$\Delta_k^* \doteq \{\zeta(k) \in \mathbb{C}|\ 0 < |\zeta(k)| < 1\} \tag{60}$$

can be conformally related by choosing the conformal factor $e^{2u}$ in (1) according to

$$e^{2u} = \frac{\left[l(\partial(\rho^2(k)))\right]^2}{4\pi^2}|\zeta(k)|^{-2\left(\frac{\theta(k)}{2\pi}\right)}. \tag{61}$$

Thus, the conical metric

$$\begin{aligned}ds_{(k)}^2 &\doteq& \frac{\left[l(\partial(\rho^2(k)))\right]^2}{4\pi^2}|\zeta(k)|^{-2\left(\frac{\varepsilon(k)}{2\pi}\right)}|d\zeta(k)|^2 = \\ &=& \frac{\left[l(\partial(\rho^2(k)))\right]^2}{4\pi^2}|\zeta(k)|^{2\left(\frac{\theta(k)}{2\pi}\right)}\frac{|d\zeta(k)|^2}{|\zeta(k)|^2},\end{aligned} \tag{62}$$

can be used to describe, as $\frac{\theta(k)}{2\pi}$ varies in $\mathbb{R}^+$, all possible conical geometries on $\Delta_k^* \subset U_{\rho^2(k)}$ including the cylindrical metric $|\phi(k)_{\rho^2(k)}|$ as a particular case corresponding to $\frac{\theta(k)}{2\pi} = 0$.
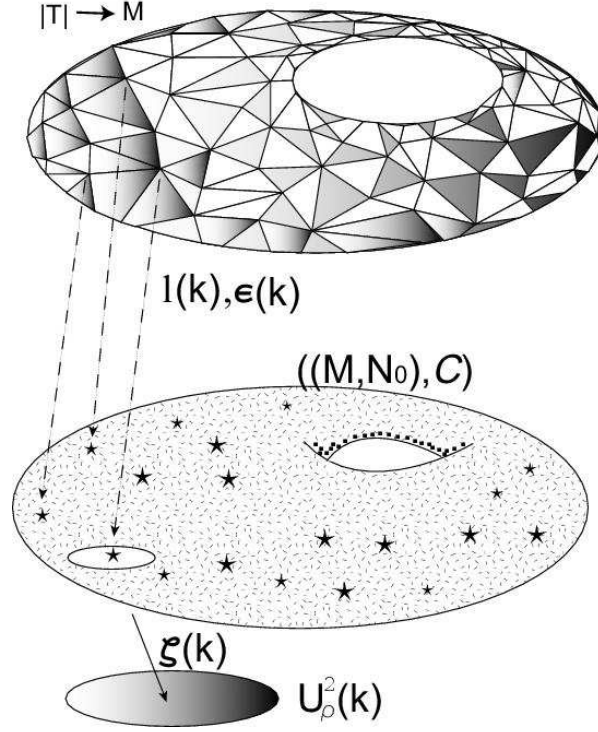
Figure 8: The decorated Riemann surface associated with a Regge triangulation.

Since $|\phi(k)_{\rho^2(k)}|$ and $ds^2_{(k)}$ correspond to the same moduli point $((M; N_0), \mathcal{C})$ $\in \mathfrak{M}_{g,N_0}$ we can naturally extend the explicit construction [8] of the mapping (45) for defining the decorated Riemann surface corresponding to a conical Regge polytope:

**Proposition 3** *Let $\{p_k\}_{k=1}^{N_0} \in M$ denote the set of punctures corresponding to the decorated vertices $\{\sigma^0(k), \frac{\varepsilon(k)}{2\pi}\}_{k=1}^{N_0}$ of the triangulation $|T_l| \to M$ and let $\Gamma$ be the ribbon graph associated with the corresponding dual conical polytope $(|P_{T_l}| \to M)$, then the map*

$$\Upsilon : (|P_{T_l}| \to M) \longrightarrow ((M; N_0), \mathcal{C}); \{\phi(k, t(k))\} \qquad (63)$$

$$\Gamma \longmapsto \bigcup_{\{\rho^0(j)\}}^{N_2(T)} U_{\rho^0(j)} \bigcup_{\{\rho^1(h)\}}^{N_1(T)} U_{\rho^1(h)} \bigcup_{\{\rho^2(k)\}}^{N_0(T)} (U_{\rho^2(k)}, \phi(k, t(k))|_{\rho^2(k)}),$$

*defines the decorated, $N_0$-pointed, Riemann surface $((M; N_0), \mathcal{C})$ canonically associated with the conical Regge polytope $|P_{T_l}| \to M$.*

In other words, through the morphism $\Upsilon$ defined by the gluing maps (55) and (56) one can generate, from a polytope barycentrically dual to a (generalized)
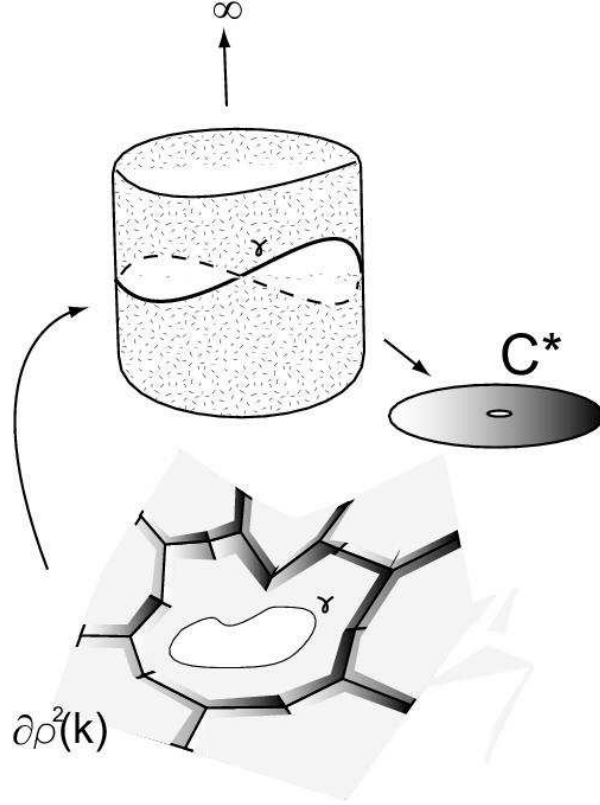
Figure 9: The cylindrical metric associated with a ribbon graph.

Regge triangulation, a Riemann surface $((M; N_0), \mathcal{C}) \in \overline{\mathfrak{M}}_{g,N_0}$ . Such a surface naturally carries the decoration provided by a choice of local coordinates $\zeta(k)$ around each puncture, of the corresponding meromorphic quadratic differential $\phi(k, t(k))|_{\rho^2(k)}$ and of the associated conical metric $ds^2_{(k)}$. It is through such a decoration that the punctured Riemann surface $((M; N_0), \mathcal{C})$ keeps track of the metric geometry of the conical Regge polytope $|P_{T_l}| \to M$ out of which $((M; N_0), \mathcal{C})$ has been generated.

The above proposition characterizes the spaces of conical Regge polytopes $\overline{RP}^{met}_{g,N_0}$ as a local covering for $\overline{\mathfrak{M}}_{g,N_0}$.

### 4.2. The Weil-Petersson geometry of Regge Polytopes.

We proceed in a similar vein and describe the Weil-Petersson geometry associated with a Regge polytope. To this end, let us start by observing that we can

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represent finer details of the geometry of $(U_{\rho^2(k)}, ds^2_{(k)})$ by opening the cone into its constituent conical sectors. To motivate such a representation, let $W_\alpha(k)$, $\alpha = 1, ..., q(k)$, be the barycenters of the edges $\sigma^1(\alpha) \in |T_l| \to M$ incident on $\sigma^0(k)$, and intercepting the boundary $\partial(\rho^2(k))$ of the polygonal cell $\rho^2(k)$. Denote by $l(\partial(\rho^2(k)))$ the length of $\partial(\rho^2(k))$, and by $\widehat{L}_\alpha(k)$ the length of the polygonal $\partial(\rho^2(k))$ between the points $W_\alpha(k)$ and $W_{\alpha+1}(k)$ (with $\alpha$ defined mod $q(k)$). In the uniformization $\zeta(k)$ of $C|lk(\sigma^0(k))|$, the points $\{W_\alpha(k)\}$ characterize a corresponding set of points on the circumference $\{\zeta(k) \in \mathbb{C} \mid |\zeta(k)| = l(\partial(\rho^2(k)))\}$, (for simplicity, we have set $\zeta_k(\sigma^0(k)) = 0$), and an associated set of $q(k)$ generators $\{\overline{W_\alpha(k)\sigma^0(k)}\}$ on the cone $(U_{\rho^2(k)}, ds^2_{(k)})$. Such generators mark $q(k)$ conical sectors

$$S_\alpha(k) \doteq \left( c_\alpha(k), \frac{l(\partial(\rho^2(k)))}{\theta(k)}, \vartheta_\alpha(k) \right), \tag{64}$$

with base

$$c_\alpha(k) \doteq \left\{ |\zeta(k)| = l(\partial(\rho^2(k))), \arg W_\alpha(k) \le \arg \zeta(k) \le \arg W_{\alpha+1}(k) \right\}, \tag{65}$$

slant radius $\frac{l(\partial(\rho^2(k)))}{\theta(k)}$, and with angular opening

$$\vartheta_\alpha(k) \doteq \frac{\widehat{L}_\alpha(k)}{l(\partial(\rho^2(k)))} \theta(k), \tag{66}$$

where $\theta(k) = 2\pi - \varepsilon(k)$ is the given conical angle. Since $\sum_{\alpha=1}^{q(k)} \vartheta_\alpha(k) = \theta(k)$, the $\{\vartheta_\alpha(k)\}$ are the representatives, in the uniformization $(U_{\rho^2(k)}, ds^2_{(k)})$, of the $q(k)$ vertex angles generating the deficit angle $\varepsilon(k)$ of $C|lk(\sigma^0(k))|$. In particular, we can formally represent the cone $(U_{\rho^2(k)}, ds^2_{(k)})$ as

$$(U_{\rho^2(k)}, ds^2_{(k)}) = \cup_{\alpha=1}^{q(k)} S_\alpha(k). \tag{67}$$

If we split open the vertex of the cone and of the associated conical sectors $S_\alpha(k)$, then the conical geometry of $(U_{\rho^2(k)}, ds^2_{(k)})$ can be equivalently described by a cylindrical strip of height $\frac{l(\partial(\rho^2(k)))}{\theta(k)}$ decorating the boundary of $\rho^2(k)$. Each sector $S_\alpha(k)$ in the cone gives rise, in such a picture, to a rectangular region in the cylindrical strip. It is profitable to explicitly represent any such a region, in the complex plane of the variable $z = x + \sqrt{-1}y$, upside down according to

$$R_{\vartheta_\alpha(k)}(k) \doteq \left\{ z \in \mathbb{C} \mid 0 \le x \le \frac{l(\partial(\rho^2(k)))}{\theta(k)}, 0 \le y \le \widehat{L}_\alpha(k) \right\}. \tag{68}$$

We can go a step further, and by means of the conformal transformation

$$W_\alpha(k) = \exp \left[ \frac{2\pi\sqrt{-1}\theta(k)}{l(\partial(\rho^2(k)))} z \right], \tag{69}$$
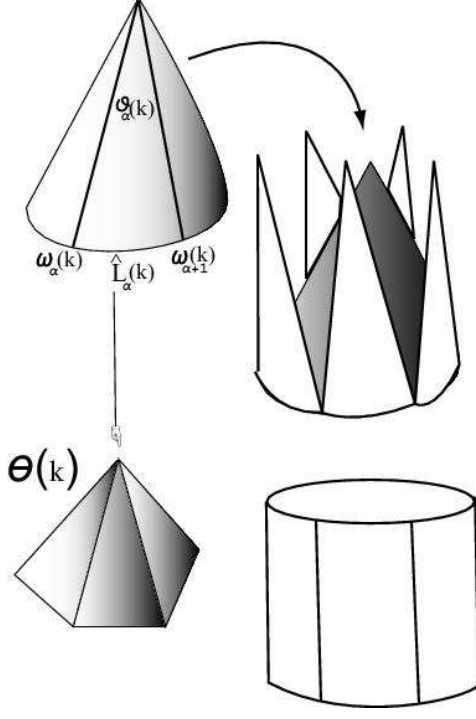
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Figure 10: The opening of the cone into its constituents conical sectors and the associated cylindrical strip.

we can map the rectangle $R_{\vartheta_\alpha(k)}(k)$ into the annulus

$$\Delta_{\vartheta_\alpha(k)}(k) \doteq \{W(k) \in \mathbb{C} | \; |t_\alpha(k)| < |W(k)| < 1\}, \tag{70}$$

where

$$|t_\alpha(k)| \doteq \exp\left[-2\pi\vartheta_\alpha(k)\right].$$

Note that

$$\frac{1}{2\pi} \ln\left(\frac{1}{|t_\alpha(k)|}\right) = \vartheta_\alpha(k), \tag{71}$$

is the modulus of $\Delta_{\vartheta_\alpha(k)}(k)$.

Such a remark further motivates the analysis of the geometry of (random) Regge triangulations from the point of view of moduli theory. In this connection note that for a given set of perimeters $\{l(\partial(\rho^2(k)))\}_{k=1}^{N_0}$ and deficit angles $\{\varepsilon(k)\}_{k=1}^{N_0}$ there are $(N_1(T) - N_0(T))$ free angles $\vartheta_\alpha(k)$. This follows by observing that, for given $l(\partial(\rho^2(k)))$ and $\varepsilon(k)$, the angles $\vartheta_\alpha(k)$ are characterized (see (66)) by the $\widehat{L}_\alpha(k)$. These latter are in a natural correspondence with the $N_1$ edges of $|P_{T_l}| \to M$, and among them we have the $N_0$ constraint $\sum_\alpha^{q(k)} \widehat{L}_\alpha(k) = l(\partial(\rho^2(k)))$. From $N_0(T) - N_1(T) + N_2(T) = 2 - 2g$,
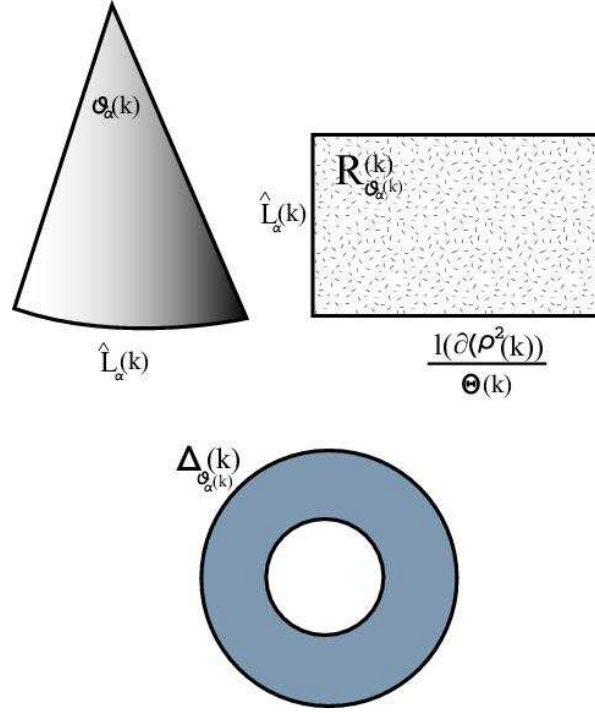
Figure 11: The mapping of a conical sector in a strip and into an anular region.

and the relation $2N_1(T) = 3N_2(T)$ associated with the trivalency, we get $N_1(T) - N_0(T) = 2N_0(T) + 6g - 6$, which exactly corresponds to the real dimension of the moduli space $\mathfrak{M}_{g,N_0}$ of genus $g$ Riemann surfaces $((M; N_0), \mathcal{C})$ with $N_0$ punctures.

According to the above procedure, we can open the cones $(U_{\rho^2(k)}, ds^2_{(k)}) = \cup_{\alpha=1}^{q(k)} S_\alpha(k)$ and map each conical sector $S_\alpha(k)$ onto a corresponding annulus

$$\Delta_{\vartheta_\alpha(k)}(k) \doteq \{W(k) \in \mathbb{C} \mid e^{-2\pi\vartheta_\alpha(k)} < |W(k)| < 1\}. \tag{72}$$

In such a setting, a natural question to discuss concerns how a deformation of the conical sectors $S_\alpha(k)$, at fixed deficit angle $\varepsilon(k)$ (and perimeter $l(\partial(\rho^2(k)))$), affects the conical geometry of $(U_{\rho^2(k)}, ds^2_{(k)})$, and how this deformation propagates to the underlying Regge polytope $|P_{T_l}| \to M$. The question is analogous to the study of the non-trivial deformations of constant curvature metrics on a surface. To this end, we adapt to our purposes a standard procedure by considering the following deformation of $S_\alpha(k)$

$$\vartheta_\alpha(k) \longmapsto \vartheta_\alpha(k)' \doteq (s+1)\vartheta_\alpha(k), \ s \in \mathbb{R}, \tag{73}$$

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and discuss its effect on the map $\Upsilon$ near the identity $s = 0$. In the annulus description of the geometry of the corresponding sector $S_\alpha(k)$, such a deformation is realized by the quasi-conformal map

$$W(k) \longmapsto f(W(k)) = W(k)\,|W(k)|^s\,, \tag{74}$$

which indeed maps the annulus (72) into the annulus

$$\{W(k) \in \mathbb{C} \mid e^{-2\pi(s+1)\vartheta_\alpha(k)} < |W(k)| < 1\}, \tag{75}$$

corresponding to a conical sector of angle $\vartheta_\alpha(k)' \doteq (s+1)\vartheta_\alpha(k)$. The study of the transformation $f(W(k))$ is well known in the modular theory of the annulus (see [12]), and goes as follows. The Beltrami differential associated with $f(\zeta(k))$ at $s = 0$, is given by

$$\frac{\partial}{\partial \overline{W(k)}}\left[\left.\frac{\partial}{\partial s}f(W(k))\right|_{s=0}\right] = \frac{W(k)}{2\overline{W(k)}}. \tag{76}$$

The corresponding Beltrami differential on (72) representing the infinitesimal deformation of $S_\alpha(k)$ in the direction $\partial/\partial(e^{-2\pi\vartheta_\alpha(k)})$ is provided by

$$
\begin{aligned}
\mu_\alpha(k) &\doteq \left(\frac{\partial e^{-2\pi(s+1)\vartheta_\alpha(k)}}{\partial s}\right)_{s=0}^{-1} \frac{\partial}{\partial \overline{W(k)}}\left[\left.\frac{\partial}{\partial s}f(W(k))\right|_{s=0}\right] = \quad (77)\\
&= -\left(\frac{1}{2\pi\vartheta_\alpha(k)}\right)e^{2\pi\vartheta_\alpha(k)}\frac{W(k)}{2\overline{W(k)}}.
\end{aligned}
$$

Recall that a Beltrami differential on the annulus is naturally paired with a quadratic differential $\widehat{\phi_\alpha(k)} \doteq C_k \frac{dW(k)\otimes dW(k)}{W^2(k)}$, $C_k$ being a real constant, via the $L^2$ inner product

$$\left\langle \mu_\alpha(k), \overline{\widehat{\phi_\alpha(k)}} \right\rangle_{\Delta_{\vartheta_\alpha(k)}(k)} \doteq \frac{\sqrt{-1}}{2}\int_{\Delta_{\vartheta_\alpha(k)}(k)} \mu(k)\overline{\widehat{\phi_\alpha(k)}}dW(k)\wedge d\overline{W(k)}. \tag{78}$$

By requiring that

$$\left\langle \mu_\alpha(k), \overline{\widehat{\phi_\alpha(k)}} \right\rangle_{\Delta_{\vartheta_\alpha(k)}(k)} = 1 \tag{79}$$

one finds the constant $C_k$ characterizing the quadratic differential $\widehat{\phi_\alpha(k)}$, dual to $\mu_\alpha(k)$, and describing the infinitesimal deformation of $S_\alpha(k)$ in the direction $d(e^{-2\pi\vartheta_\alpha(k)})$, (i.e., a cotangent vector to $\mathfrak{M}_{g,N_0}$ at $((M;N_0),\mathcal{C});\{ds^2_{(k)}\}))$. A direct computation provides

$$\widehat{\phi_\alpha(k)} = -\frac{e^{-2\pi\vartheta_\alpha(k)}}{\pi}\frac{dW(k)\otimes dW(k)}{W^2(k)}. \tag{80}$$

Note that whereas $\phi(k)_{\rho^2(k)}$ may be thought of as a cotangent vector to one of the $\mathbb{R}^{N_0}$ fiber of $\mathfrak{M}_{g,N_0}\times\mathbb{R}^N_+$, the quadratic differential $\widehat{\phi_\alpha(k)}$projects down
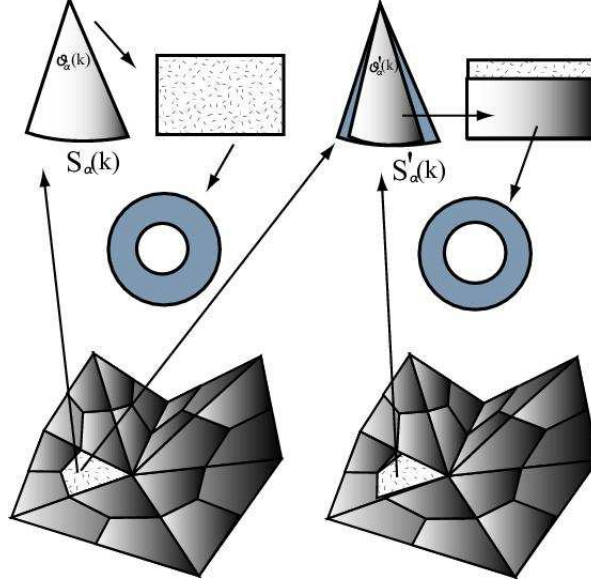
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Figure 12: The modular deformation of a conical sector in a Regge polytope.

to a deformation in the base $\mathfrak{M}_{g,N_0}$, and as such is a much more interesting object than $\phi(k)_{\rho^2(k)}$. The modular nature of $\widehat{\phi_\alpha(k)}$ comes clearly to the fore if, along the lines of section 3.3, we compute its associated Weil-Petersson norm according to

$$\left\|\widehat{\phi_\alpha(k)}\right\|_{W-P} = \int_{\Delta_{\vartheta_\alpha(k)}(k)} \frac{|\widehat{\phi_\alpha(k)}|^2}{g_{hyp}}, \tag{81}$$

where $g_{hyp}$ denotes the hyperbolic metric

$$g_{hyp} \doteq \left(\frac{\pi^2}{\ln^2 |t_\alpha(k)|}\right) \frac{dW(k)d\overline{W(k)}}{|W(k)|^2 \sin^2\left(\pi \frac{\ln |W(k)|}{\ln |t_\alpha(k)|}\right)} \tag{82}$$

on the annulus $\Delta_{\vartheta_\alpha(k)}(k)$, and where $|t_\alpha(k)| \doteq e^{-2\pi\vartheta_\alpha(k)}$. A direct computation provides [12]

$$\left\|\widehat{\phi_\alpha(k)}\right\|_{W-P} = |t_\alpha(k)|^2 \left(\frac{1}{\pi} \ln \frac{1}{|t_\alpha(k)|}\right)^3 = 8\vartheta_\alpha^3(k)e^{-4\pi\vartheta_\alpha(k)}. \tag{83}$$

According to proposition 3, the decorated Riemann surface

$$(((M; N_0), \mathcal{C}); \{ds_{(k)}^2\})$$

is generated by glueing the oriented domains $(U_{\rho^2(k)}, ds_{(k)}^2)$ along the ribbon graph $\Gamma$ associated with the Regge polytope $|P_{T_l}| \to M$. We can indipendently

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and holomorphically open the distinct cones $\{(U_{\rho^2(k)}, ds^2_{(k)})\}$ on corresponding sectors $\{S_\alpha(k)\}$, then it follows that to any such $(((M; N_0), \mathcal{C}); \{ds^2_{(k)}\})$ we can associate a well defined Weil-Petersson metric which can be immediately read off from (83). Since there are $6g - 6 + 2N_0$ independent conical sectors $S_\alpha(k)$ on the Riemann surface $(((M; N_0), \mathcal{C}); \{ds^2_{(k)}\})$ corresponding to the Regge polytope $|P_{T_l}| \to M$, (at fixed deficit angles $\{\varepsilon(k)\}$ and perimeters $\{l(\partial(\rho^2(k)))\}$), we can find a corresponding sequence of angles $\{\vartheta_\alpha(k)\}$ and relabel them as $\{\vartheta_H\}_{H=1}^{6g-6+2N_0}$. If we define

$$\tau_H \doteq e^{-2\pi\vartheta_H} \tag{84}$$

then we can immediately write the Weil-Petersson metric $ds^2_{W-P}$ associated with the Regge polytope $|P_{T_l}| \to M$

$$
\begin{aligned}
ds^2_{W-P}(|P_{T_l}|) &= \sum_{H=1}^{6g-6+2N_0} \frac{2\pi^3 d\tau_H^2}{\tau_H^2 \left(\ln\frac{1}{\tau_H}\right)^3} = \\
&= \pi^2 \sum_{H=1}^{6g-6+2N_0} \frac{d\vartheta_H^2}{\vartheta_H^3}.
\end{aligned}
\tag{85}
$$

If we respectively denote by $\theta(H)$ and $l(H)$ the (fixed) conical angle $\theta(k)$ and the perimeter $l(\partial(\rho^2(k)))$ corresponding to the modular variable $\vartheta_H$, (note that the same $\theta(H)$ and $l(H)$ correspond to the distinct $\{\vartheta_H\}$ which are incident on the same vertex), then according to (66) we can rewrite (85) as

$$ds^2_{W-P}(|P_{T_l}|) = \sum_{H=1}^{6g-6+2N_0} \frac{\pi^2}{\theta(H)} \left(\frac{\widehat{L}_H}{l(H)}\right)^{-3} d\left(\frac{\widehat{L}_H}{l(H)}\right) \otimes d\left(\frac{\widehat{L}_H}{l(H)}\right), \tag{86}$$

where $\widehat{L}_H$ is the length variable $\widehat{L}_\alpha(k)$ associated with $\vartheta_H$.

Note that (85) and (86) show a singular behavior when $\vartheta_H \to 0$ (or equivalently when $\widehat{L}_H \to 0$), such a behavior occurs for instance when the Regge triangulation (and the corresponding polytope) degenerates and exhibits vertices where thinner and thinner triangles are incident. As recalled in paragraph 2.3, this is usually considered a pathology of Regge calculus. However, in view of the modular correspondence we have described in this paper, it simply corresponds to the well-known incompleteness (with respect of the complex structure of the moduli space) of the Weil-Petersson metric on $\mathfrak{M}_{g,N_0}$ as we approach the boundary (the compactifying divisor) of $\mathfrak{M}_{g,N_0}$ in $\overline{\mathfrak{M}}_{g,N_0}$. In such a sense, as already remarked in the introductory remarks, such a pathology of Regge triangulations has a modular meaning and is not accidental.

### 4.3. The W-P volume on the space of Regge polytopes.

With the metric $ds^2_{W-P}(|P_{T_l}|)$ we can associate a well defined volume form

$$\Omega_{W-P}(|P_{T_l}|) = \tag{87}$$

$$= \left[ \det \left( \frac{\pi^2}{\theta(H)} \left( \frac{\widehat{L}_H}{l(H)} \right)^{-3} \right) \right]^{\frac{1}{2}} d \left( \frac{\widehat{L}_1}{l(1)} \right) \bigwedge \cdots \bigwedge d \left( \frac{\widehat{L}_{6g-6+2N_0}}{l(6g-6+2N_0)} \right),$$

which can be rewritten as a power of product of 2-forms of the type $d \left( \frac{\widehat{L}_H}{l(H)} \right) \wedge$
$d \left( \frac{\widehat{L}_{H+1}}{l(H+1)} \right)$, (this being connected with the Kähler form associated with (86)). Such forms are directly related with the Chern classes (47) of the line bundles $\mathcal{CL}_k$ and play a distinguished role in the Kontsevich-Witten model. The possibility of expressing the Weil-Petersson Kähler form in terms of the Chern classes (47) is a deep and basic fact of the geometry of $\overline{\mathfrak{M}}_{g,N_0}$, and it is a pleasant feature of the model discussed here that such a connection can be motivated by rather elementary considerations.

The Weil-Petersson volume form $\Omega_{W-P}(|P_{T_l}|)$ allows us to integrate over the space

$$RP^{met}_{g,N_0}(\{\varepsilon(H)\}, \{l(H)\}) \tag{88}$$

of distinct Regge polytopes $|P_{T_l}| \to M$ with given deficit angles $\{\varepsilon(k)\}$ and perimeters $\{l(\partial(\rho^2(k)))\}$. To put such an integration in a proper perspective and give it a suggestive physical meaning we shall explicitly consider the set of Regge polytopes

$$RP^{met}_{g,N_0}(\{q(H)\}) \doteq \left\{ |P_{T_l}| \to M | \; \varepsilon(H) = 2\pi - \frac{\pi}{3}q(H); \; l(H) = \frac{\sqrt{3}}{3}aq(H) \right\}, \tag{89}$$

which contain the equilateral Regge polytopes dual to dynamical triangulations. According to proposition 3, $RP^{met}_{g,N_0}(\{q(H)\})$ is a combinatorial description of $\mathfrak{M}_{g,N_0}$, and the volume form $\Omega_{W-P}(|P_{T_l}|)$ can be considered as the pull-back under the morphism $\Upsilon$, (see (63)) of the Weil-Petersson volume form $\omega^{3g-3+N_0}_{WP}/(3g-3+N_0)!$ on $\mathfrak{M}_{g,N_0}$.

Stated differently, the integration of $\Omega_{W-P}(|P_{T_l}|)$ over the space (89) is equivalent to the Weil-Petersson volume of $\mathfrak{M}_{g,N_0}$. It is well-known that the Weil-Petersson form $\omega_{WP}$ extends (as a (1,1) current) to the compactification $\overline{\mathfrak{M}}_{g,N_0}$, and, if we denote by $\overline{RP}^{met}_{g,N_0}(\{q(H)\})$ the compactified orbifold associated with $RP^{met}_{g,N_0}(\{q(H)\})$, then we can write

$$\frac{1}{N_0!} \int_{\overline{RP}^{met}_{g,N_0}(\{q(H)\})} \Omega_{W-P}(|P_{T_l}|) =$$

$$= \frac{1}{N_0!} \int_{\overline{\mathfrak{M}}_{g,N_0}} \frac{\omega^{3g-3+N_0}_{WP}}{(3g-3+N_0)!} = Vol \left( \overline{\mathfrak{M}}_{g,N_0} \right) \tag{90}$$

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where we have divided by $N_0(T)!$ in order to factor out the labelling of the $N_0(T)$ punctures. Since $\overline{RP}_{g,N_0}^{met}(\{q(H)\})$ is a (smooth) stratified orbifold acted upon by the automorphism group $Aut_\partial(P_{T_a})$, we can explicitly write the left side of (90) as an orbifold integration over the distinct orbicells $\Omega_{T_a}$ (27) in which $\overline{RP}_{g,N_0}^{met}(\{q(H)\})$ is stratified

$$\frac{1}{N_0!} \sum_{T \in \mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]} \frac{1}{|Aut_\partial(P_{T_a})|} \int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \Omega_{W-P}(|P_{T_l}|) = \qquad (91)$$

$$= VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right),$$

(the orbifold integration is defined in [14], Th. 3.2.1), where the summation is over all distinct dynamical triangulations with given unlabeled curvature assignments weighted by the order $|Aut_\partial(P_T)|$ of the automorphisms group of the corresponding dual polytope. The relation (91) provides a non-trivial connection between dynamical triangulations (labelling the strata of $\overline{RP}_{g,N_0}^{met}(\{q(H)\})$ (or, equivalently, of $\overline{\mathfrak{M}}_{g,N_0}$), and the fixed connectivity Regge triangulations in each strata $\Omega_{T_a}$. Some aspects of this relation have already been discussed by us elsewhere, [15]. Here we will exploit (91) in order to directly relate $VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)$ to the partition function of 2D simplicial quantum gravity. To this end let us sum both members of (91) over the set of all possible curvature assignments $\{q(H)\}$ on the $N_0$ unlabelled vertices of the triangulations, and note that

$$Card\left[\mathcal{DT}(N_0)\right] \doteq \sum_{\mathcal{DT}(N_0)} \frac{1}{|Aut_\partial(P_{T_a})|} =$$

$$= \frac{1}{N_0!} \sum_{\{q(H)\}_{H=1}^{N_0}} \sum_{T \in \mathcal{DT}[\{q(H)\}_{H=1}^{N_0}]} \frac{1}{|Aut_\partial(P_{T_a})|}$$

provides the number of distinct (generalized) dynamical triangulations with $N_0$ unlabelled vertices. Since $VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)$ does not depend of the curvature assignments $\{q(H)\}$, from (91) we get

$$\frac{1}{N_0!} \sum_{\{q(H)\}_{H=1}^{N_0}} \sum_{T \in \mathcal{DT}[\{q(i)\}_{i=1}^{N_0}]} \frac{1}{|Aut_\partial(P_{T_a})|} \int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \Omega_{W-P}(|P_{T_l}|) = \quad (92)$$

$$= \sum_{\mathcal{DT}(N_0)} \frac{1}{|Aut_\partial(P_{T_a})|} \int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \Omega_{W-P}(|P_{T_l}|) =$$

$$= (Card\{q(H)\}) \, VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right),$$

where $Card\{q(H)\}$ denotes the number of possible curvature assignments on the $N_0$ unlabelled vertices of the triangulations. Dividing both members by

$(Card\{q(H)\})$, we eventually get the relation

$$\sum_{\mathcal{DT}(N_0)} \frac{1}{|Aut_\partial(P_{T_a})|} \int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}} = VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right), \qquad (93)$$

(the number $Card\{q(H)\}$ has been shifted under the integral sign for typographical convenience). We have the following

**Lemma 4** *For $N_0$ sufficiently large there exist a positive constant $K(g)$ independent from $N_0$ but possibly dependent on the genus $g$, and a positive constant $\kappa$ independent both from $N_0$ and $g$ such that*

$$\int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}} \simeq K(g)e^{-\kappa N_0}. \qquad (94)$$

In order to prove this result let us start by recalling that the large $N_0(T)$ asymptotics of the triangulation counting $Card\left[\mathcal{DT}[N_0]\right]$ can be obtained from purely combinatorial (and matrix theory) arguments [8], [17] to the effect that

$$Card\left[\mathcal{DT}[N_0]\right] \sim \frac{16c_g}{3\sqrt{2\pi}} \cdot e^{\mu_0 N_0(T)} N_0(T)^{\frac{5g-7}{2}}\left(1 + O(\frac{1}{N_0})\right), \qquad (95)$$

where $c_g$ is a numerical constant depending only on the genus $g$, and $e^{\mu_0} = (108\sqrt{3})$ is a (non-universal) parameter depending on the set of triangulations considered (here the generalized triangulations, barycentrically dual to trivalent graphs; in the case of regular triangulations in place of $108\sqrt{3}$ we would get $e^{\mu_0} = (\frac{4^4}{3^3})$). Thus, if we denote by

$$\left\langle \int \frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}} \right\rangle_{\mathcal{DT}[N_0]} \doteq \qquad (96)$$

$$= \frac{1}{Card\left[\mathcal{DT}[N_0]\right]} \sum_{\mathcal{DT}(N_0)} \frac{1}{|Aut_\partial(P_{T_a})|} \int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}}$$

the average value of $\int \Omega_{W-P}(|P_{T_l}|)/Card\{q(H)\}$ over the set $\mathcal{DT}[N_0]$, (dropping the integration range $\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})$ for notational ease), then we can write the large $N_0$ asymptotics of the left side member of (91) as

$$\sum_{\mathcal{DT}(N_0)} \frac{1}{|Aut_\partial(P_{T_a})|} \int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})} \frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}} \simeq \qquad (97)$$

$$\simeq \frac{16c_g}{3\sqrt{2\pi}} \left\langle \int \frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}} \right\rangle_{\mathcal{DT}[N_0]} e^{\mu_0 N_0(T)} N_0(T)^{\frac{5g-7}{2}}\left(1 + O(\frac{1}{N_0})\right).$$

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On the other hand, from the Manin-Zograf asymptotic analysis of $VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right)$ for fixed genus $g$ and large $N_0$, we have[2][18]

$$VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right) = \tag{98}$$

$$= \pi^{2(3g-3+N_0)}(N_0+1)^{\frac{5g-7}{2}}C^{-N_0}\left(B_g + \sum_{k=1}^{\infty}\frac{B_{g,k}}{(N_0+1)^k}\right),$$

where $C = -\frac{1}{2}j_0\frac{d}{dz}J_0(z)|_{z=j_0}$, ($J_0(z)$ the

Bessel function, $j_0$ its first positive zero); (note that $C \simeq 0.625....$). The genus dependent parameters $B_g$ are explicitly given [18] by

$$\begin{cases} B_0 = \frac{1}{A^{1/2}\Gamma(-\frac{1}{2})C^{1/2}}, & B_1 = \frac{1}{48}, \\ B_g = \frac{A^{\frac{g-1}{2}}}{2^{2g-2}(3g-3)!\Gamma(\frac{5g-5}{2})C^{\frac{5g-5}{2}}}\left\langle\tau_2^{3g-3}\right\rangle, & g \geq 2 \end{cases} \tag{99}$$

where $A \doteq -j_0^{-1}J_0'(j_0)$, and $\left\langle\tau_2^{3g-3}\right\rangle$ is a Kontsevich-Witten intersection number [13][19], (the coefficients $B_{g,k}$ can be computed similarly-see [18] for details). By Inserting (98) in the left hand side of (91) and by taking into account (97) we eventually get

$$\frac{16c_g}{3\sqrt{2\pi}}\left\langle\int\frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}}\right\rangle_{\mathcal{DT}[N_0]}e^{\mu_0 N_0(T)}N_0(T)^{\frac{5g-7}{2}}\left(1 + O(\frac{1}{N_0})\right) = \tag{100}$$

$$= \pi^{2(3g-3+N_0)}(N_0+1)^{\frac{5g-7}{2}}C^{-N_0}\left(B_g + \sum_{k=1}^{\infty}\frac{B_{g,k}}{(N_0+1)^k}\right),$$

which by direct comparison provides

$$\left\langle\int\frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}}\right\rangle_{\mathcal{DT}[N_0]} \simeq \frac{3\sqrt{2\pi}B_g\pi^{2(3g-3+N_0)}}{16c_g}e^{(|\ln C|-\mu_0)N_0}. \tag{101}$$

This proves the lemma with

$$K(g) = \frac{3\sqrt{2\pi}B_g\pi^{2(3g-3)}}{16c_g}, \tag{102}$$

and

$$\kappa = \mu_0 - |\ln C| - 2\ln\pi \approx 2.472. \tag{103}$$

Thus, according to the above lemma, we can rewrite the left member of (93) as

$$VOL\left(\overline{\mathfrak{M}}_{g,N_0}\right) = \sum_{\mathcal{DT}(N_0)}\frac{1}{|Aut_\partial(P_{T_a})|}\int_{\Omega_{T_a}(\{q(k)\}_{k=1}^{N_0})}\frac{\Omega_{W-P}(|P_{T_l}|)}{Card\{q(H)\}} = \tag{104}$$

$$= K(g)\sum_{\mathcal{DT}(N_0)}\frac{1}{|Aut_\partial(P_{T_a})|}e^{-\kappa N_0},$$

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which has the structure of the canonical partition function for dynamical triangulation theory [9]. Roughly speaking, we may interpret (104) by saying that dynamical triangulations count the (orbi)cells which contribute to the volume (each cell containing the Regge triangulations whose adjacency matrix is that one of the dynamical triangulation which label the cell), whereas integration over the cells weights the fluctuations due to all Regge triangulations which, in a sense, represent the deformational degrees of freedom of the given dynamical triangulation labelling the cell.

### 4.4. The Hodge-Deligne decomposition.

Further properties of triangulated surfaces have their origin in the Hodge geometry of the corresponding Riemann surface. To this end, let us consider the collection of $N_0$ meromorphic 1-forms associated with the quadratic differentials $\{\phi(k)\}$,

$$\left(\sqrt{\phi(1)}, ..., \sqrt{\phi(N_0)}\right) \in \Omega^1(\{p_j\}), \tag{105}$$

where $\{p_j\}$ is the divisor of $\sqrt{\phi}$, and $\Omega^1(\{p_j\})$ denotes the space of meromorphic 1-forms $\vartheta$ with divisor $(D_\vartheta) \leq \{p_j\}$. In each uniformization $\zeta(k)$ we have

$$\sqrt{\phi(k)} \doteq \frac{\sqrt{-1}}{2\pi} \frac{l(\partial(\rho^2(k)))}{\zeta(k)} d\zeta(k) \underset{DT}{=} \frac{\sqrt{-1}}{2\pi} \frac{\left(\frac{\sqrt{3}}{3}a\right) q(k)}{\zeta(k)} d\zeta(k), \tag{106}$$

where the subscript $DT$ refers to dynamical triangulations. According to (57) the residues of $\{\sqrt{\phi(k)}\}$ provide the perimeters $\{l(\partial(\rho^2(k)))\}$. However, the role of $\{\sqrt{\phi(k)}\}$ is more properly seen in connection with the introduction, on the cohomology group $H^1(M, N_0; \mathbb{C}) = H^1(M_k; \mathbb{R}) \bigotimes_{\mathbb{R}} \mathbb{C}$, of a Hodge structure analogous to the classical Hodge decomposition of $H^h(M; \mathbb{C})$ generated by the spaces $\mathcal{H}^{r, h-r}$ of harmonic $h$-forms on $M$ of type $(r, h - r)$. Such a decomposition does not hold, as it stands, for punctured surfaces (since $H^1(M, N_0; \mathbb{C})$ can be odd-dimensional), but it can be replaced by a mixed Deligne-Hodge decomposition such that

$$H^1(M, N_0; \mathbb{C}) \doteq \bigoplus_{p,r} I^{p,r}, \tag{107}$$

where

$$I^{1,1} = \Omega^1(\{p_j\}) \cap \overline{\Omega^1(\{p_j\})}, \qquad I^{1,0} = H^{1,0}(M), \qquad I^{0,1} = H^{0,1}(M) \tag{108}$$

with $\dim_C I^{1,1} = N_0 - 1$, (this dimensionality is strictly related with the constraint (7)), and where $H^{1,0}(M) \simeq H^0(M, \Omega^1)$, ($H^{0,1}(M) \simeq \overline{H^0(M, \Omega^1)}$ ), denotes the space of holomorphic 1-forms on $M$. More explicitly, the subspace $I^{1,1}$ can be characterized as

$$I^{1,1} \cap H^1(M, N_0; \mathbb{R}) = \left\{\sqrt{-1}\frac{\partial h}{\partial z}dz \mid h \in \mathcal{H}^0(M, N_0)\right\}, \tag{109}$$

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where $\mathcal{H}^0(M, N_0)$ denotes the space of real-valued harmonic functions on $M$ which have at worst logarithmic singularities along the $N_0$ punctures [20]. It follows that the forms $\{\sqrt{\phi(k)}\}_{k=1}^{N_0}$ are in $I^{1,1}$ (at the level of cohomology, *i.e.* up to exact forms), for we can write

$$\sqrt{\phi(k)} - \frac{\sqrt{-1}}{2\pi}l(\partial(\rho^2(k)))d\ln|\zeta(k)| = \tag{110}$$

$$= \frac{\sqrt{-1}}{4\pi}l(\partial(\rho^2(k)))\left(\frac{d\zeta(k)}{\zeta(k)} - \frac{d\overline{\zeta}(k)}{\overline{\zeta}(k)}\right) \in \Omega^1(\{p_j\}) \cap \overline{\Omega^1(\{p_j\})}.$$

Therefore, if we fix a puncture, say $p_{N_0}$, and consider the $N_0 - 1$ cohomology classes

$$\vartheta_k \doteq \frac{1}{2\pi\sqrt{-1}}\left[\frac{d\zeta(N_0)}{\zeta(N_0)} - \frac{d\zeta(k)}{\zeta(k)} - d\ln\frac{|\zeta(N_0)|}{|\zeta(k)|}\right], \tag{111}$$

then $\{\vartheta_k\}_{k=1}^{N_0-1}$ defines a basis for $I^{1,1}$ in terms of the complex coordinates $\{\zeta(k)\}_{k=1}^{N_0}$ uniformizying $((M; N_0), \mathcal{C})$ around the punctures. At the level of cohomology, we can equivalently write

$$\vartheta_k = \left[\frac{\sqrt{\phi(k)}}{l(\partial(\rho^2(k)))} - \frac{\sqrt{\phi(N_0)}}{l(\partial(\rho^2(N_0)))}\right]. \tag{112}$$

The basis $\{\vartheta_k\}_{k=1}^{N_0-1}$ can be completed to $H^1(M, N_0; \mathbb{C})$ by introducing also a basis $\{\varphi(a)\}_{a=1}^g$ for $H^{1,0}(M)$, ($g$ being the genus of $M$) according to

$$\int_M \varphi_a \wedge \overline{\varphi}_b = -i\delta_{ab}. \tag{113}$$

Note that

$$\int_M \vartheta_j \wedge \overline{\varphi}_a = 0, \tag{114}$$

since $\alpha \in I^{1,1}$ iff $\int_M \alpha \wedge \overline{\varphi}_b = 0$. It follows that on $H^1(M, N_0; \mathbb{R})$ we have the (Hodge-Deligne) inner product

$$(\alpha, \beta)_{H-D} \doteq \sum_{p \in \{p_k\}_{k=1}^{N_0}} RES_p(\alpha)RES_p(\beta) + \int_M *\alpha \wedge \beta \tag{115}$$

for the (unique) harmonic representatives $\alpha$ and $\beta$ of elements $[\alpha]$ and $[\beta]$ in $H^1(M, N_0; \mathbb{R})$, where $*$ denotes the Hodge dual and $RES_p(\cdot)$ is the residue map. By extending such an inner product to a hermitian form on $H^1(M, N_0; \mathbb{C})$ one gets a (mixed Hodge) metric on $H^1(M, N_0; \mathbb{C})$ [21] [22]. It is worth stressing that the bilinear form (115) induces the non-degenerate bilinear forms

$$(\alpha, \beta)_{(H-D)}^{\parallel} \doteq \sum_{p \in \{p_k\}_{k=1}^{N_0}} RES_p(\alpha)RES_p(\beta), \text{ on } \frac{H^1(M, N_0; \mathbb{R})}{H^1(M, \mathbb{R})} \tag{116}$$

and

$$(\alpha, \beta)_{(H-D)}^{\perp} \doteq \int_M *\alpha \wedge \beta, \text{ on } H^1(M, \mathbb{R}). \tag{117}$$

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### 4.5. A regularized Dirichlet norm.

In a similar vein, let us consider the function $v$ defining the conformal class of the conical geometry (1) around the generic puncture $p_k$, *i.e.*,

$$v|_{U_{\rho^2(k)}} = -\frac{\varepsilon(k)}{2\pi} \ln |\zeta(k)| + u, \tag{118}$$

where the function $u$ is continuous and $C^2$ on $U_{\rho^2(k)} - p_k$ and is such that, for $\zeta(k) \to 0$, $|\zeta(k)| \frac{\partial u}{\partial \zeta(k)}$, and $|\zeta(k)| \frac{\partial u}{\partial \overline{\zeta}(k)}$ both $\to 0$, (see (1)). Since

$$*dv|_{U_{\rho^2(k)}} = -\sqrt{-1}\left(\frac{\partial v}{\partial \zeta(k)} d\zeta(k) - \frac{\partial v}{\partial \overline{\zeta}(k)} d\overline{\zeta}(k)\right) = \tag{119}$$

$$= -\pi \left(\frac{\varepsilon(k)}{2\pi}\right)\left(\frac{d\zeta(k)}{2\pi\sqrt{-1}\zeta(k)} - \frac{d\overline{\zeta}(k)}{2\pi\sqrt{-1}\overline{\zeta}(k)}\right) + *du,$$

it follows that the polar part of $*dv$ is in $I^{1,1}$ (in general the whole $*dv$ is not in $I^{1,1}$ since $\partial\overline{\partial}u \neq 0$) and we can consider its Hodge-Deligne norm on $\frac{H^1(M,N_0;\mathbb{R})}{H^1(M,\mathbb{R})}$

$$(*dv, *dv)_{H-D}^{\parallel} = \sum_{p \in \{p_k\}_{k=1}^{N_0}} RES_p\,(*dv)\,RES_p\,(*dv) = 4\pi^2 \sum_{k=1}^{N_0} \left(\frac{\varepsilon(k)}{2\pi}\right)^2, \tag{120}$$

(the factor $4\pi^2$ is inserted for later convenience). This latter expression is directly related with a natural regularization of the otherwise ill-defined Dirichlet's energy associated with $v$, (*i.e.*, the $L^2$ norm of the gradient of $v$). Let $M_\varrho \doteq \cup_{k=1}^{N_0}\Delta_k^\varrho$ where $\Delta_k^\varrho \doteq \{\zeta(k) \in \mathbb{C}|\, \varrho \leq |\zeta(k)| \leq 1\}$, (thus $M_\varrho$ is topologically $M$ with $N_0$ disks $c_\varrho(k)$ of radius $\varrho$ around the punctures removed). Since $\overline{\partial}(\frac{d\zeta(k)}{\zeta(k)}) = 0$ in $\Delta_k^\varrho$, we have

$$\frac{\sqrt{-1}}{2}\int_{M_\varrho} \frac{\partial v}{\partial \zeta}\frac{\partial v}{\partial \overline{\zeta}} d\zeta \wedge d\overline{\zeta} = \frac{1}{4}\int_{M_\varrho} dv \wedge *dv = \tag{121}$$

$$= \frac{1}{4}\int_{M_\varrho} du \wedge *du + \frac{1}{2}\sum_{k=1}^{N_0}\left(-\frac{\varepsilon(k)}{2\pi}\right)\frac{\sqrt{-1}}{2}\int_{\Delta_k^\varrho \cap M_\varrho}\left(\partial u \wedge \frac{d\overline{\zeta}(k)}{\overline{\zeta}(k)} - \overline{\partial} u \wedge \frac{d\zeta(k)}{\zeta(k)}\right)$$

$$+ \frac{1}{4}\sum_{k=1}^{N_0}\left(\frac{\varepsilon(k)}{2\pi}\right)^2 \frac{\sqrt{-1}}{2}\int_{\Delta_k^\varrho \cap M_\varrho}\frac{d\zeta(k) \wedge d\overline{\zeta}(k)}{|\zeta(k)|^2} =$$

$$= \frac{1}{4}\int_{M_\varrho} du \wedge *du - \frac{1}{2}\sum_{k=1}^{N_0}\left(\frac{\varepsilon(k)}{2\pi}\right)\frac{\sqrt{-1}}{2}\oint_{\partial c_\varrho(k)} u\left(\frac{d\overline{\zeta}(k)}{\overline{\zeta}(k)} - \frac{d\zeta(k)}{\zeta(k)}\right) -$$

$$- \frac{\pi}{2}\ln\varrho \sum_{k=1}^{N_0}\left(\frac{\varepsilon(k)}{2\pi}\right)^2,$$

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where $\partial c_\varrho(k)$ is the circle (with positive orientation) of radius $\varrho$ around the generic puncture $p_k$. Thus, we get

$$\frac{\sqrt{-1}}{2} \int_{M_\varrho} \frac{\partial v}{\partial \zeta} \frac{\partial v}{\partial \overline{\zeta}} d\zeta \wedge d\overline{\zeta} + \frac{\pi}{2} \ln \varrho \sum_{k=1}^{N_0} \left( \frac{\varepsilon(k)}{2\pi} \right)^2 = \tag{122}$$

$$\frac{\sqrt{-1}}{2} \int_{M_\varrho} \frac{\partial v}{\partial \zeta} \frac{\partial v}{\partial \zeta} d\zeta \wedge d\overline{\zeta} + \frac{1}{8\pi} \ln \varrho \, (*dv, *dv)_{H-D}^{\parallel} =$$

$$= \frac{1}{4} \int_{M_\varrho} du \wedge *du - \frac{1}{2} \sum_{k=1}^{N_0} \left( \frac{\varepsilon(k)}{2\pi} \right) \frac{\sqrt{-1}}{2} \oint_{\partial c_\varrho(k)} u \left( \frac{d\overline{\zeta}(k)}{\overline{\zeta}(k)} - \frac{d\zeta(k)}{\zeta(k)} \right).$$

Since the right hand side of this expression is well-defined in the $\varrho \to 0^+$ limit, we can define a regularized $L^2$-norm of $dv$ according to

$$\int_M^{reg} dv \wedge *dv \doteq \lim_{\varrho \to 0^+} \left[ \int_{M_\varrho} dv \wedge *dv + \frac{1}{2\pi} \ln \varrho \, (*dv, *dv)_{H-D}^{\parallel} \right]. \tag{123}$$

We can explicitly rewrite $\int_M^{reg} dv \wedge *dv$ as the integral over $M$ of a $(1,1)$ current by noticing that

$$\frac{1}{2} \sum_{k=1}^{N_0} \left( \frac{\varepsilon(k)}{2\pi} \right) \frac{\sqrt{-1}}{2} \oint_{\partial c_\varrho(k)} u \left( \frac{d\overline{\zeta}(k)}{\overline{\zeta}(k)} - \frac{d\zeta(k)}{\zeta(k)} \right) = \tag{124}$$

$$= -\frac{\pi}{2} \sum_{k=1}^{N_0} \left( \frac{\varepsilon(k)}{2\pi} \right) \int_{c_\varrho(k)} u \left( \partial \left( \frac{d\overline{\zeta}(k)}{2\pi\sqrt{-1}\overline{\zeta}(k)} \right) - \overline{\partial} \left( \frac{d\zeta(k)}{2\pi\sqrt{-1}\zeta(k)} \right) \right) +$$

$$+ \frac{1}{2} \sum_{k=1}^{N_0} \left( \frac{\varepsilon(k)}{2\pi} \right) \frac{\sqrt{-1}}{2} \int_{c_\varrho(k)} \partial u \wedge \left( \frac{d\overline{\zeta}(k)}{\overline{\zeta}(k)} \right) - \overline{\partial} u \wedge \left( \frac{d\zeta(k)}{\zeta(k)} \right) =$$

$$= \pi \sum_{k=1}^{N_0} \left( \frac{\varepsilon(k)}{2\pi} \right) \int_M u \delta_{p_k} \frac{\sqrt{-1}}{2} d\zeta(k) \wedge d\overline{\zeta}(k) +$$

$$+ \frac{1}{2} \sum_{k=1}^{N_0} \left( \frac{\varepsilon(k)}{2\pi} \right) \frac{\sqrt{-1}}{2} \int_{c_\varrho(k)} \partial u \wedge \left( \frac{d\overline{\zeta}(k)}{\overline{\zeta}(k)} \right) - \overline{\partial} u \wedge \left( \frac{d\zeta(k)}{\zeta(k)} \right).$$

Under the stated hypotheses on $u$, the integrals over the disks $c_\varrho(k)$ go uniformly to 0 as $\varrho \to 0^+$, and we can write

$$\int_M^{reg} dv \wedge *dv = \int_M du \wedge *du + 4\pi \sum_{k=1}^{N_0} \left( -\frac{\varepsilon(k)}{2\pi} \right) \int_M u \delta_{p_k} \frac{\sqrt{-1}}{2} d\zeta(k) \wedge d\overline{\zeta}(k). \tag{125}$$

More generally, if $v_1$ and $v_2$ are two locally summable functions on $((M; N_0), \mathcal{C})$ with local behavior

$$v_1|_{U_{\rho^2(k)}} = -\frac{\varepsilon_1(k)}{2\pi} \ln |\zeta(k)| + u_1, \tag{126}$$

$$v_2|_{U_{\rho^2(k)}} = -\frac{\varepsilon_2(k)}{2\pi} \ln |\zeta(k)| + u_2,$$

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then we can define a regularized inner product of their gradients according to

$$\int_M^{reg} dv_1 \wedge *dv_2 \doteq \lim_{\varrho \to 0^+} \left[ \int_{M_\varrho} dv_1 \wedge *dv_2 + \frac{1}{2\pi} \ln \varrho \, (*dv_1, *dv_2)^{\|}_{H-D} \right] = \quad (127)$$

$$= \int_M du_1 \wedge *du_2 + 2\pi \sum_{k=1}^{N_0} \int_M \left( -\frac{\varepsilon_1(k)}{2\pi} u_2 - \frac{\varepsilon_2(k)}{2\pi} u_1 \right) \delta_{p_k} \frac{\sqrt{-1}}{2} d\zeta(k) \wedge d\bar{\zeta}(k).$$

### 4.6. The regularized Liouville action.

The conformal class of the metric representing the Regge triangulation associated with $((M; N_0), \mathcal{C})$ is given by $ds^2 = e^{2v} ds_0^2$ where $ds_0^2$ is a smooth metric on $M$ and where the conformal factor $v$ around the generic puncture $p_k$, is provided by (118). The Gaussian curvature $K$ of $ds^2$ is related to the Gaussian curvature $K_0$ of the smooth metric $ds_0^2$ by the relation

$$K dA = K_0 dA_0 - d * dv, \qquad (128)$$

where $dA_0$ and $dA$ are the area elements of $ds_0^2$ and $ds^2$, respectively. By interpreting $d * dv$ as a $(1, 1)$ current, a formal direct computation yields

$$d * dv = \pi \sum_{k=1}^{N_0} \left( -\frac{\varepsilon(k)}{2\pi} \right) \left[ \bar{\partial} \left( \frac{d\zeta(k)}{2\pi \sqrt{-1} \zeta(k)} \right) - \partial \left( \frac{d\bar{\zeta}(k)}{2\pi \sqrt{-1} \bar{\zeta}(k)} \right) \right] + 2\sqrt{-1} \partial \bar{\partial} u,$$

$$(129)$$

$$= -2\pi \sum_{k=1}^{N_0} \left( -\frac{\varepsilon(k)}{2\pi} \right) \frac{\sqrt{-1}}{2} \delta_{p_k} d\zeta(k) \wedge d\bar{\zeta}(k) + 2\sqrt{-1} \partial \bar{\partial} u.$$

From which we get the (inhomogeneous) Liouville equation associated with the conformal metric $ds^2$ with conical singularities $\{-\frac{\varepsilon(k)}{2\pi}\}$, *i.e.*

$$-4\partial_\zeta \partial_{\bar{\zeta}} u = e^{2v} K - K_0 - 2\pi \sum_{k=1}^{N_0} \left( -\frac{\varepsilon(k)}{2\pi} \right) \delta_{p_k}. \qquad (130)$$

Note that, upon integrating (130) over $M$ we have

$$\frac{1}{2\pi} \int_M e^{2v} K dA_0 = \frac{1}{2\pi} \int_M K_0 dA_0 + \sum_{k=1}^{N_0} \left( -\frac{\varepsilon(k)}{2\pi} \right), \qquad (131)$$

which, since $\frac{1}{2\pi} \int_M K_0 dA_0 = \chi(M)$ and $\sum_{k=1}^{N_0} \left( -\frac{\varepsilon(k)}{2\pi} \right) = -\chi(M)$ -(see (7)), is a restatement of the vanishing of the Euler class (10) of $((M; N_0), \mathcal{C})$.

From the definition of regularized inner product (127) it is easily verified that formally (130) can be obtained as the Euler-Lagrange equation of the functional

$$S(v|\{\varepsilon(h)\}) \doteq \frac{1}{4} \int_M^{reg} dv \wedge *dv - \int_M e^{2v} K dA_0 + 2 \int_M v K_0 dA_0, \qquad (132)$$

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by considering variations $v_t$ of the form $v_t = v + th$, where the function $h$ is continuous and $C^2$ on each $U_{\rho^2(k)} - p_k$ and is such that, for $\zeta(k) \to 0$, $|\zeta(k)| \frac{\partial h}{\partial \zeta(k)}$, and $|\zeta(k)| \frac{\partial h}{\partial \overline{\zeta}(k)}$ both $\to 0$. The functional $S(v|K, K_0)$ is basically a compact rewriting of the Takhtajan and Zograf regularized Liouville action [23].

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